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Ordered Regular Proper Semigroups

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INTRODUCTION

This paper is in the line of our systematical study of ordered semigroups. By an ordered semigroup we mean a semigroup with a simple order which is compatible with the semigroup operation. In [6] we characterized ordered idempotent semigroups, in [7] we determined all types of subsemigroups generated by an inverse pair in an ordered semigroup, in [8] we characterized ordered completely regular semigroups, and in [9] we characterized some kind of ordered inverse semigroups which we called proper.

In the algebraic theory of semigroups regular semigroups constitute an important type of semigroups which is a generalization of inverse semigroups. But until now very little was known about regular semigroups. As the first step of the study of ordered regular semigroups, the main purpose of this note is to characterize ordered regular proper semigroups. The results and the method of this paper are similar, to some extent, to those in our previous paper [9], but are much more complicated.

As a by-product, in Section 2 we define proper semigroups purely-algebraically, whereas defined in [9] for ordered inverse semigroups, and give some important properties of regular proper semigroups.

In Section 3 we give a theorem asserting that each $\mathscr{D}$-class of an ordered semigroup belongs to one of the two types, which we call L-type and R-type (Theorem 3.1). This theorem, which is a generalization of a theorem given in [6], is so fundamental that it seems to play an important role in the algebraic characterization of orderability of regular semigroups.

Sections 4 and 5 we devote to the study of the characterization of ordered regular proper semigroups.

1. PRELIMINARIES

The notations of Clifford and Preston [2] are used throughout. In particular, if ρ is a congruence on a semigroup S, then S/ρ denotes the factor semigroup of S modulo ρ, and $\rho^{\natural}$ denotes the natural homomorphism of S onto S/ρ.

We always denote by S a semigroup and by E the set of idempotents of a semigroup S, unless otherwise mentioned.

By the *minimum group congruence* on S we mean the smallest congruence σ on S for which S/σ is a group. Unless otherwise mentioned we always denote by σ the minimum group congruence of S.

If $A \subseteq S$ and there exists a $\mathscr{D}$-class containing A, then the $\mathscr{D}$-class is denoted by $D(A)$. When $A = \{a\}$ we write $D(a)$ in place of $D(\{a\})$. In other words, for $a \in S$, $D(a)$ is the $\mathscr{D}$-class which contains the element a. If A is itself a $\mathscr{D}$-class of S, then clearly $D(A) = A$.

If E is a subsemigroup of S, then E itself as a semigroup is decomposed into $\mathscr{D}$-classes. A $\mathscr{D}$-class of the semigroup E is called a $\mathscr{D}_E$-class, and $D(A)$ and $D(a)$ of the semigroup E are denoted by $D_E(A)$ and $D_E(a)$, respectively. For $e, f \in E$, if $D_E(e) = D_E(f)$, then we write $e\mathscr{D}_E f$.

If S is an idempotent semigroup, then S is a semilattice of rectangular bands ([*2*]; Exercise 1, Section 4.2), and it is easily seen that every rectangular band which is a constituent of the decomposition is a $\mathscr{D}$-class of S. Thus the set of $\mathscr{D}$-classes of S can be considered to form a semilattice which is called the *associated semilattice* of S and is denoted by S^*. We denote the semilattice operation in S^* by $\circ$ and the order relation in S^* by $\preccurlyeq$. In particular, for a, $b \in S$, we have $D(a) \circ D(b) = D(ab) = D(ba) = D(b) \circ D(a)$.

S is called an *ordered semigroup* if S is a semigroup and a simply-ordered set at the same time and satisfies the condition that, if $a, b, c \in S$ and $a \leqslant b$, then $ac \leqslant bc$ and $ca \leqslant cb$. In an ordered semigroup S we say that c *lies between a and b* if either $a \leqslant c \leqslant b$ or $b \leqslant c \leqslant a$. An element a of S is called *positive* if $a < a^2$, while a is called *negative* if $a > a^2$. For $a \in S$ the number of distinct natural powers of a is called the *order* of a. Evidently a is of order 1 if and only if a is an idempotent. For a subset T of S we denote by max T and min T the greatest element and the least element of T, respectively.

Now we list some results from our previous papers which will be needed in the following discussions.

LEMMA 1.1 ([*6*], Lemma 2). *In an ordered idempotent semigroup S, both ab and ba lie between a and b.*

LEMMA 1.2 ([*6*], Lemma 3). *In an ordered idempotent semigroup S, if c lies between a and b, then $ab = acb$.*

LEMMA 1.3 ([*6*], Lemma 4). *In an ordered idempotent semigroup S, if c lies between a and b, then $D(c) \succcurlyeq D(a) \circ D(b)$.*

LEMMA 1.4 ([*6*], Theorem 1). *In an ordered idempotent semigroup S, each $\mathscr{D}$-class consists of either only one $\mathscr{L}$-class or only one $\mathscr{R}$-class.*

By Lemma 1.4, a $\mathscr{D}$-class in an ordered idempotent semigroup S belongs to one of two types. A $\mathscr{D}$-class which consists of only one $\mathscr{L}$-class is called a *$\mathscr{D}$-class of L-type*, while a $\mathscr{D}$-class which consists of only one $\mathscr{R}$-class is called a *$\mathscr{D}$-class of R-type*.

LEMMA 1.5 ([6], Theorem 2). *In an ordered idempotent semigroup S, let $a \leqslant b$. If the $\mathscr{D}$-class $D(a) \circ D(b)$ is of L-type, then*

$$ab = \min \{y; y \in D(a) \circ D(b) \text{ and } a \leqslant y\},$$

$$ba = \max \{y; y \in D(a) \circ D(b) \text{ and } y \leqslant b\}.$$

If $D(a) \circ D(b)$ is of R-type, then

$$ab = \max \{y; y \in D(a) \circ D(b) \text{ and } y \leqslant b\},$$

$$ba = \min \{y; y \in D(a) \circ D(b) \text{ and } a \leqslant y\}.$$

LEMMA 1.6 ([6], Theorem 3). *Let F, G, H be $\mathscr{D}$-classes of an ordered idempotent semigroup S such that $F \preccurlyeq H$ and $G \preccurlyeq H$. Then F and G are comparable in the associated semilattice S^*.*

LEMMA 1.7 ([7], Corollary of Lemma 1). *In an ordered semigroup S, E is, if it is nonvoid, a subsemigroup of S.*

LEMMA 1.8 ([7], Theorem 1). *In an ordered semigroup, let x' be an inverse of x. Then x is an idempotent if and only if x' is an idempotent.*

LEMMA 1.9 ([7], Lemma 5). *In an ordered semigroup, let x' and y' be inverses of x and y, respectively. Then $y'x'$ is an inverse of xy.*

LEMMA 1.10 ([7], Lemma 6). *In an ordered semigroup, let x' be an inverse of x. If $x' \leqslant x$, then x' is nonpositive and x is nonnegative.*

LEMMA 1.11 ([7], Corollary of Lemma 7). *In an ordered semigroup, let x' be an inverse of x. Then x is an element of order 2 if and only if x' is of order 2.*

LEMMA 1.12 ([7], Equation (8)). *In an ordered semigroup, let x' be an inverse of x. If both x and x' are elements of order 2 and $x' \leqslant x$, then $x'^2 < x' < x'x < x < x^2$.*

LEMMA 1.13 ([7], Theorem 2). *In an ordered regular semigroup, an element of finite order can have order only 1 or 2.*

2. Regular Proper Semigroups

In this section we study regular proper semigroups. Except for the last part, all results are purely-algebraic.

Let S be a semigroup. A subset U of S is called *left-unitary* if $ux \in U$ and $u \in U$ imply $x \in U$. A *right-unitary* subset is defined dually. A subset of S which is both left and right-unitary is simply called *unitary*. If the set E of idempotents forms a unitary subsemigroup of S, then S is called *proper*. Here we give two lemmas which are due to Howie and Lallement.

Lemma 2.1 ([4], Lemma 2.1). *For a regular semigroup S, the following conditions are equivalent to each other:*

(1) *S is proper,*

(2) *E is a left-unitary subset of S.*

Lemma 2.2 ([4], Lemma 2.2). *In a regular proper semigroup S, $ab \in E$ if and only if $ba \in E$.*

Now we give some properties of proper semigroups.

Lemma 2.3. *In a proper semigroup S, let x' be an inverse of an element x. Then x is an idempotent if and only if x' is an idempotent.*

Proof. If x is an idempotent, then $x \in E$ and $xx' \in E$. Hence we have $x' \in E$. This proves the "only if" part of the lemma. We can prove the "if" part in a similar way.

Lemma 2.4. *In a proper semigroup S, let x' and y' be inverses of x and y, respectively. Then $y'x'$ is an inverse of xy.*

Proof. Since yy', $x'x \in E$, we have $x'xyy' \in E$, and so

$$xy = xx'xyy'y = x(x'xyy')^2 y = xx'xyy'x'xyy'y = xyy'x'xy.$$

Similarly we have $y'x' = y'x'xyy'x'$.

Lemma 2.5. *In a regular proper semigroup S, let x' be an inverse of x. Then $xzx' \in E$ if and only if $z \in E$.*

Proof. Since S is proper, $z \in E$ is equivalent to $x'xz \in E$ and, by Lemma 2.2, $x'xz \in E$ is equivalent to $xzx' \in E$.

Lemma 2.6. *In a regular proper semigroup S, the following relations for $x, y \in S$ are equivalent to one another:*

(1) *$xy' \in E$ for some inverse y' of y,*
(2) *$y'x \in E$ for some inverse y' of y,*
(3) *$x'y \in E$ for some inverse x' of x,*
(4) *$yx' \in E$ for some inverse x' of x,*
(5) *$xy' \in E$ for every inverse y' of y,*
(6) *$y'x \in E$ for every inverse y' of y,*
(7) *$x'y \in E$ for every inverse x' of x,*
(8) *$yx' \in E$ for every inverse x' of x,*
(9) *$xey' \in E$ for some inverse y' of y and some $e \in E$,*
(10) *$y'ex \in E$ for some inverse y' of y and some $e \in E$,*
(11) *$x'ey \in E$ for some inverse x' of x and some $e \in E$,*
(12) *$yex' \in E$ for some inverse x' of x and some $e \in E$,*
(13) *$xe = fy$ for some $e, f \in E$,*
(14) *$ex = yf$ for some $e, f \in E$,*
(15) *$exf = eyf$ for some $e, f \in E$,*
(16) *$uxv = uyv$ for some $u, v \in S^1$,*
(17) *$exe = eye$ for some $e \in E$.*

Proof. (1) is equivalent to (2) by Lemma 2.2. Similarly, (3) is equivalent to (4). Now (2) implies (7). In fact, by Lemma 2.5, we have $y'xx'y \in E$ and so, since $y'x \in E$, we have $x'y \in E$. Similarly, (1) implies (8), (3) implies (6), and (4) implies (5). It is trivial that (5), (6), (7) and (8) imply (1), (2), (3) and (4), respectively. Thus the first eight conditions are equivalent to one another. (1) implies (9), since $xx'xy' = xy' \in E$ and $x'x \in E$. (9) implies (13), since $x(ey'y) = (xey')y$. (13) implies (15). In fact, since $xe = fy$, we have $fxe = f(xe)e = f(fy)e = fye$. (15) implies (16) trivially. (16) implies (15). If $u, v \in S$, then we take inverses u' and v' of u and v, respectively. Putting $u'u = e$ and $vv' = f$, we have $exf = u'uxvv' = u'uyvv' = eyf$. If u or v is an empty symbol, then, taking an arbitrary idempotent for e or f, we can proceed as above. (15) implies (17). Since $exf = eyf$, we have $fexfe = feyfe$ and $fe \in E$. (17) implies (2). By Lemma 2.4, $ey'e$ is an inverse of $eye = exe$. Hence $exey'e = (exe)(ey'e) \in E$ and so, since S is proper, we have $xey' \in E$. Therefore, by Lemma 2.2, we have $ey'x \in E$ and so $y'x \in E$. Thus the conditions (1)-(9), (13), (15), (16) and (17) are equivalent to one another. The remaining assertions of the lemma can be proved similarly.

LEMMA 2.7. *In a regular proper semigroup S, we put $x\rho y$ if and only if one of the relations in Lemma 2.6 holds for $x, y \in S$. Then ρ is the minimum group congruence σ on S and E constitutes a σ-class.*

Proof. First we show that ρ is a congruence relation. For every $x \in S$ we have $x\rho x$, since $xx' \in E$. If $x\rho y$, then $xy' \in E$ and so, by (4) of Lemma 2.6, we have $y\rho x$. If $x\rho y$ and $y\rho z$, then $xy', yz' \in E$ and so $xy'yz' \in E$. Hence, by

(9) of Lemma 2.6, we have $x\rho z$. Thus ρ is an equivalence relation. If $x\rho y$, then $y'x \in E$ and $xy' \in E$. Hence $zz'y'x \in E$ and so, by Lemma 2.2, we have $xzz'y' \in E$. Also, by Lemma 2.5, we have $zxy'z' \in E$. But, by Lemma 2.4, $z'y'$ and $y'z'$ are inverses of yz and zy, respectively. Hence we have $xz\rho yz$ and $zx\rho zy$. Thus ρ is a congruence relation. Next we show that E constitutes a ρ-class. If $e \in E$ and $x\rho e$, then, by definition, we have $ex \in E$. Since S is proper, we have $x \in E$. On the other hand, if $e, f \in E$, then $ef \in E$ and so $e\rho f$. Hence E constitutes a ρ-class. Now we take an inverse x' of x. Then $(xx')\,x = x(x'x) = x$, $xx' \in E$, and $x'x \in E$. Hence, in the factor semigroup S/ρ, $E\rho^{\natural}$ is the identity and $x'\rho^{\natural}$ is an inverse of $x\rho^{\natural}$. Thus S/ρ is a group. Finally, let τ be a congruence on S such that S/τ is a group. Then, for $e \in E$, we have $(e\tau^{\natural})^2 = e\tau^{\natural}$ and so $e\tau^{\natural}$ is the identity of S/τ. Let $x\rho y$. Then, by definition, $exe = eye$ for some $e \in E$. Hence $x\tau^{\natural} = (exe)\,\tau^{\natural} = (eye)\,\tau^{\natural} = y\tau^{\natural}$ and so $x\tau y$. Therefore $\rho \subseteq \tau$. Thus ρ is the minimum group congruence σ and E constitutes a σ-class.

LEMMA 2.8. *In a regular proper semigroup S, the following relations for $x, y \in S$ are equivalent to one another:*

(1) *y is an inverse of x,*

(2) *$R_x \cap L_y$ contains an idempotent and $(x\sigma^{\natural})\,(y\sigma^{\natural}) = 1$,*

(3) *$R_y \cap L_x$ contains an idempotent and $(x\sigma^{\natural})\,(y\sigma^{\natural}) = 1$.*

Proof. (1) implies (2). In fact, we have $xy \in E$, $xy\mathscr{R}x$, $xy\mathscr{L}y$ and $(x\sigma^{\natural})\,(y\sigma^{\natural}) = (xy)\,\sigma^{\natural} = 1$. (2) implies (3). By Theorem 2.17 of [2], we have $yx \in R_y \cap L_x$. Moreover, $(yx)\,\sigma^{\natural} = (y\sigma^{\natural})\,(x\sigma^{\natural}) = 1$ and so, by Lemma 2.7, we have $yx \in E$. (3) implies (1). In fact, we can prove similarly that $e = xy \in R_x \cap L_y$ and $e \in E$. Hence $x = ex = xyx$ and $y = ye = yxy$.

THEOREM 2.9. *For a regular semigroup S, the following conditions are equivalent to each other:*

(1) *S is proper,*

(2) *E constitutes a σ-class.*

Proof. By Lemma 2.7, (1) implies (2). Next we suppose that E constitutes a σ-class and that $ex \in E$ *and* $e \in E$. Then $x\sigma^{\natural} = (x\sigma^{\natural})\,(e\sigma^{\natural}) = (xe)\,\sigma^{\natural} = 1$. Hence $x \in E$ and so E is a left-unitary subset of S. Therefore, by Lemma 2.1, S is proper.

Remark. In [9], a proper ordered inverse semigroup was defined to be an ordered inverse semigroup S in which E constitutes a σ'-class where, for $x, y \in S$, $x\sigma'y$ if and only if $ex = ey$ for some $e \in E$. It was shown in [5], or is easily seen by Lemmas 2.6 and 2.7, that in an inverse semigroup S, σ' is the minimum group congruence. Hence, by Theorem 2.9, the notion of proper semigroups defined in this section is a generalization of that of proper ordered inverse semigroups defined in [9].

In the rest of this section, we are concerned with ordered semigroups.

LEMMA 2.10. *In an ordered regular proper semigroup S, each σ-class is a convex subset of S.*

Proof. Let $x\sigma y$ and $x \leqslant z \leqslant y$. Then we have $exe = eye$ for some $e \in E$. Hence $exe \leqslant eze \leqslant eye = exe$ and so $exe = eze$. Therefore we have $x\sigma z$.

THEOREM 2.11. *For an ordered regular semigroup S, the following two conditions are equivalent to each other:*

(1) *S is proper,*

(2) *E is a convex subset of S.*

Proof. First we suppose that S is proper. Then, by Lemma 2.7, E is a σ-class and so, by Lemma 2.10, E is a convex subset of S. Conversely, we suppose that E is convex in S and that $ex \in E$ and $e \in E$. Let x' be an inverse of x and first we suppose that $x' \leqslant x$. Now we assume that x is not an idempotent. Then, by Lemmas 1.8 and 1.10, x is positive and x' is negative. Hence we have $xx'x = x < x^2$ and $x'^2 < x' = x'xx'$ and so $x' < xx' < x$ and $xx' \in E$. But, by Lemma 1.9, $x'e$ is an inverse of ex and so, by Lemma 1.8, we have $x'e \in E$. Since E is convex we have $x'xx' = x' < x'e$ and $ex < x = xx'x$. Hence we have $xx' < e$ and $e < xx'$ at the same time, which is absurd. In the case when $x \leqslant x'$ we can prove $x \in E$ similarly. Hence E is a left-unitary subset of S and so, by Lemma 2.1, S is proper.

COROLLARY 2.12. *In an ordered regular proper semigroup S, there are no elements of finite order except idempotents.*

Proof. Let a be an element of finite order. Then, by Lemma 1.13, a has order 1 or 2. Now we assume that a is an element of order 2. Let a' be an inverse of a. Then, by Lemma 1.11, a' is also an element of order 2. Hence, by Lemma 1.12, if $a' \leqslant a$, then $a'a < a < a^2$, and, if $a \leqslant a'$, then $a^2 < a < aa'$. Since a^2, $a'a$, $aa' \in E$, in both cases E is not convex in S, which contradicts Theorem 2.11. Thus a is an element of order 1, that is, an idempotent.

3. REGULAR $\mathscr{D}$-CLASSES IN AN ORDERED SEMIGROUP

In this section, we give some properties of regular $\mathscr{D}$-classes in an ordered semigroup.

THEOREM 3.1. *Let D be a $\mathscr{D}$-class of an ordered semigroup S. Then either*

(1) *every $\mathscr{L}$-class in D contains at most one idempotent, or*

(2) *every $\mathscr{R}$-class in D contains at most one idempotent.*

Proof. We assume that the conclusion of the theorem is false. Then there exist an $\mathscr{L}$-class L and an $\mathscr{R}$-class R which contain two distinct idempotents $e < f$ and $g < h$, respectively. Let $x \in L \cap R$ and let s and t be inverses of x in $R_e \cap L_h$ and $R_f \cap L_g$, respectively. These inverses really exist, by Theorem 2.18 of [2]. Now we have $sx \in E$, $sx \in R_s \cap L_x = H_e$, and so, by Lemma 2.15 of [2], we have $sx = e$. Similarly, we have $tx = f$, $xt = g$, and $xs = h$. Hence $sx = e < f = tx$, $xt = g < h = xs$ and so, we have $s < t$ and $t < s$ at the same time, which is absurd.

A $\mathscr{D}$-class in which every $\mathscr{L}$-class contains at most one idempotent is called a *$\mathscr{D}$-class of R-type*. A *$\mathscr{D}$-class of L-type* is defined dually. Theorem 3.1 shows that every $\mathscr{D}$-class is either of R-type or of L-type. Of course, a $\mathscr{D}$-class may be of R-type and at the same time of L-type.

Now let S be an ordered semigroup and we suppose that E is nonvoid. Then, by Lemma 1.7, E is a subsemigroup of S. Thus we can consider $\mathscr{D}_E$-equivalence. Moreover, for $e, f \in E$, if $e\mathscr{D}_E f$, then evidently $e\mathscr{D}f$. Hence, for a $\mathscr{D}_E$-class F, we can consider $D(F)$. In particular, if S is an ordered idempotent semigroup, then clearly $E = S$ and $D(F) = F$.

LEMMA 3.2. *Let D be a $\mathscr{D}$-class of an ordered idempotent semigroup S. Then D is of L-type in the sense of Section 1 if and only if D is of L-type in the sense of this section. Also D is of R-type in the sense of Section 1 if and only if D is of R-type in the sense of this section.*

Proof. We prove only the assertion for the L-type since the case of the R-type can be proved similarly. First we suppose that D is of L-type in the sense of Section 1. Let $e, f \in D$ and $e\mathscr{R}f$. Then, by assumption, we have $e\mathscr{L}f$ and so $e\mathscr{H}f$. Hence, by Lemma 2.15 of [2], we have $e = f$. Conversely, we suppose that D is of L-type in the sense of this section. Let $e, f \in D$. Then there exists an element g such that $e\mathscr{R}g\mathscr{L}f$. By assumption we have $e = g$ and so $e\mathscr{L}f$.

Lemma 3.2 assures that for an ordered idempotent semigroup the two definitions of L-type and R-type cause no confusion.

In an ordered semigroup S, as we mentioned above, for $e, f \in E$, $e\mathscr{D}_E f$ implies $e\mathscr{D}f$, but the converse is not always true. As for $\mathscr{R}$ and $\mathscr{L}$, the situation is different. For $e, f \in E$, $e\mathscr{R}f$ in E if and only if $e\mathscr{R}f$ in S, and $e\mathscr{L}f$ in E if and only if $e\mathscr{L}f$ in S ([7], p. 265).

LEMMA 3.3. *Let F be a $\mathscr{D}_E$-class of an ordered semigroup S. If $D(F)$ is a*

$\mathscr{D}$-class of L-type, then F is a $\mathscr{D}_E$-class of L-type. Also, if D(F) is of R-type, then F is of R-type.

Proof. The proof is evident.

LEMMA 3.4. *Let D be a regular $\mathscr{D}$-class of an ordered semigroup S. If D is of L-type, then all inverses of an element $x \in D$ lie in the same $\mathscr{L}$-class. If D is of R-type, then all inverses of an element $x \in D$ lie in the same $\mathscr{R}$-class.*

Proof. Let x' and x'' be inverses of the same element x in a regular $\mathscr{D}$-class D of L-type. Then $xx'\mathscr{R}x\mathscr{R}xx''$ and, since D is of L-type, we have $xx' = xx''$. Hence $x'\mathscr{L}xx' = xx''\mathscr{L}x''$. The second assertion can be proved similarly.

4. STRUCTURE THEORY OF ORDERED REGULAR PROPER SEMIGROUPS

In this section we study the structure of an ordered regular proper semigroup. Throughout this section we denote by S an ordered regular proper semigroup. Since E is nonvoid, E is by Lemma 1.7, a subsemigroup of S. Hence we can consider the associated semilattice E^* of the idempotent semigroup E, which is constituted by the set of all $\mathscr{D}_E$-classes. By Lemma 2.7, there exists the minimum group congruence σ on S. We denote the group S/σ by Γ. By Lemma 2.10 we can introduce an order in Γ in a natural way; that is, for $\alpha, \beta \in \Gamma$, we define $\alpha \leqslant \beta$ if and only if there exist elements a and b of S such that $a\sigma^\natural = \alpha$, $b\sigma^\natural = \beta$ and $a \leqslant b$. It is easily seen that with this order Γ turns out to be a simply-ordered group. In the rest of this section we assume that the group Γ is ordered with this order. The identity element of Γ is denoted by 1. An element a of S is called an *element of L-type* or *of R-type* depending if the $\mathscr{D}$-class which contains a is of L-type or of R-type.

THEOREM 4.1. *Let D be a $\mathscr{D}$-class of an ordered regular proper semigroup S and let $a, b \in D$;*

(1) *if D is of L-type, $a\mathscr{R}b$ and $a\sigma^\natural = b\sigma^\natural$, then $a = b$,*

(2) *if D is of R-type, $a\mathscr{L}b$ and $a\sigma^\natural = b\sigma^\natural$, then $a = b$.*

Proof. We prove (1) only. Let a' be an inverse of a and put $aa' = e$, $a'a = f$, $ba' = g$, and $a'b = h$. Then we have $e \in E$, $e \in R_a \cap L_{a'} = R_b \cap L_{a'}$, and $(b\sigma^\natural)(a'\sigma^\natural) = (a\sigma^\natural)(a'\sigma^\natural) = (aa')\sigma^\natural = 1$. Hence, by Lemma 2.8, a' is an inverse of b. Therefore $g, h \in E$, $h\mathscr{R}a'\mathscr{R}f$, and $g\mathscr{R}b\mathscr{R}a\mathscr{R}e$. Since D is of L-type we have $h = f$ and $g = e$. Hence

$$a = aa'a = ea = ga = ba'a = bf = bh = ba'b = b.$$

For $e \in E$, we denote

$$\Gamma(e) = \{a\sigma^\natural;\ a\mathscr{R}e\}.$$

LEMMA 4.2. *In an ordered regular proper semigroup S, if $e, f \in E$ and $e\mathscr{D}_E f$, then $\Gamma(e) = \Gamma(f)$.*

Proof. We denote the $\mathscr{D}_E$-class which contains e and f by F. If F is of R-type, then $e\mathscr{R}f$ and so $a\mathscr{R}e$ is equivalent to $a\mathscr{R}f$. Hence $\Gamma(e) = \Gamma(f)$. Next we suppose that F is of L-type. For every $\alpha \in \Gamma(e)$ there exists an element a such that $a\sigma^\natural = \alpha$ and $a\mathscr{R}e$. Since F is of L-type we have $e\mathscr{L}f$ and so $fe = f$. Hence $fa\mathscr{R}fe = f$ and $(fa)\sigma^\natural = (f\sigma^\natural)(a\sigma^\natural) = a\sigma^\natural = \alpha$. Therefore $\Gamma(e) \subseteq \Gamma(f)$. Similarly, $\Gamma(f) \subseteq \Gamma(e)$ and so $\Gamma(e) = \Gamma(f)$.

For $F \in E^*$, we put

$$\Gamma(F) = \Gamma(e)$$

where e is an element of F. By Lemma 4.2, $\Gamma(F)$ is well-defined irrespective of the choice of $e \in F$.

LEMMA 4.3. *In an ordered regular proper semigroup S, let $e\mathscr{D}_E f$, $e\mathscr{R}a$, $f\mathscr{R}b$, $a\sigma^\natural = b\sigma^\natural$ and let a' and b' be inverses of a and b, respectively. Then $a'a\mathscr{D}_E b'b$.*

Proof. We denote by F the $\mathscr{D}_E$-class which contains e and f, and by C the $\mathscr{D}$-class in S which contains F. First we suppose that the $\mathscr{D}$-class C is of R-type. Then, by Lemma 3.3, the $\mathscr{D}_E$-class F is of R-type and so $a\mathscr{R}e\mathscr{R}f\mathscr{R}b$. Hence

$$aa' \in E, \qquad aa' \in R_a \cap L_{a'} = R_b \cap L_{a'},$$
$$(b\sigma^\natural)(a'\sigma^\natural) = (a\sigma^\natural)(a'\sigma^\natural) = (aa')\sigma^\natural = 1$$

and so, by Lemma 2.8, a' is an inverse of b. Therefore, by Lemma 3.4, we have $a'\mathscr{R}b'$ and so $a'a\mathscr{R}a'\mathscr{R}b'\mathscr{R}b'b$. Thus we have $a'a\mathscr{D}_E b'b$. Next we suppose that C is of L-type. Then $e\mathscr{R}a\mathscr{R}aa'$ and so $e = aa'$. Hence, by Lemma 3.3, we have $f\mathscr{L}e = aa'\mathscr{L}a'$ and so $f \in R_b \cap L_{a'}$. Moreover,

$$(b\sigma^\natural)(a'\sigma^\natural) = (a\sigma^\natural)(a'\sigma^\natural) = (aa')\sigma^\natural = 1.$$

Hence, by Lemma 2.8, b is an inverse of a'. Therefore, by Lemma 3.4, we have $a\mathscr{L}b$ and so $a'a\mathscr{L}a\mathscr{L}b\mathscr{L}b'b$. Thus $a'a\mathscr{D}_E b'b$.

Let $F \in E^*$ and let $\alpha \in \Gamma$ such that $\alpha \in \Gamma(F)$. We first choose $e \in F$, then an element $a \in S$ such that $e\mathscr{R}a$ and $a\sigma^\natural = \alpha$ (such an element a really exists, since $\alpha \in \Gamma(F)$), and then an inverse a' of a. By Lemma 4.3, the $\mathscr{D}_E$-class which contains $a'a$ is determined only by F and α, irrespective of the choice of elements e, a and a'. This $\mathscr{D}_E$-class is denoted by F^α. For $F \in E^*$,

$$\Delta(F) = \{F^\alpha; \alpha \in \Gamma(F)\}$$

is called the *derived family* determined by F.

LEMMA 4.4. *In an ordered regular proper semigroup S, let F, $G \in E^*$. Then $G \in \Delta(F)$ if and only if $G \subseteq D(F)$.*

Proof. First we suppose $G \in \Delta(F)$. Then $G = F^\alpha$ for some $\alpha \in \Gamma(F)$. Hence, by definition, $e \in F$, $e\mathscr{R}a$, a' is an inverse of a, and $a'a \in G$. Since $e\mathscr{R}a\mathscr{L}a'a$ we have $a'a \in D(F)$ and so $G \subseteq D(F)$. Conversely, we suppose that $G \subseteq D(F)$. We take $f \in F$ and $g \in G$ arbitrarily. Since $f, g \in D(F)$ there exists an element a such that $f\mathscr{R}a\mathscr{L}g$. We put $a\sigma^\natural = \alpha$. Then, by definition, $\alpha \in \Gamma(F)$. Moreover, by Theorem 2.18 of [2], $R_g \cap L_f$ contains an inverse a' of a. Since $a'a \in R_{a'} \cap L_a = R_g \cap L_g = H_g$ we have, by Lemma 2.15 of [2], $a'a = g$. Hence, by definition, $G = F^\alpha \in \Delta(F)$.

LEMMA 4.5. *In an ordered regular proper semigroup S, suppose that $F, G \in E^*$, $G \preccurlyeq F$ and that a' is an inverse of a, $a\sigma^\natural = \alpha$, $aa' \in F$, and $g \in G$. Then $\alpha \in \Gamma(G)$ and $a'ga \in G^\alpha$.*

Proof. We have $gaa' \in G \circ F = G$, $gaa'\mathscr{R}ga$, and

$$(ga)\,\sigma^\natural = (g\sigma^\natural)\,(a\sigma^\natural) = a\sigma^\natural = \alpha.$$

Hence $\alpha \in \Gamma(G)$. Moreover, by Lemma 1.9, $a'g$ is an inverse of ga and so, by definition, $a'ga = (a'g)\,(ga) \in G^\alpha$.

THEOREM 4.6. *In an ordered regular proper semigroup S,*

(1) *for each $F \in E^*$, either all $\mathscr{D}_E$-classes in $\Delta(F)$ are of L-type or all $\mathscr{D}_E$-classes in $\Delta(F)$ are of R-type,*

(2) $\Gamma = \bigcup_{F\in E^*} \Gamma(F)$,

(3) *for every $F \in E^*$, we have $1 \in \Gamma(F)$ and $F^1 = F$,*

(4) *for F, $G \in E^*$, if $G \preccurlyeq F$ and $\alpha \in \Gamma(F)$, then $\alpha \in \Gamma(G)$ and $G^\alpha \preccurlyeq F^\alpha$,*

(5) *for $F \in E^*$, if $\alpha \in \Gamma(F)$ and $\beta \in \Gamma(F^\alpha)$, then $\alpha\beta \in \Gamma(F)$ and $F^{\alpha\beta} = (F^\alpha)^\beta$,*

(6) *for $F \in E^*$, if $\alpha \in \Gamma(F)$, then $\alpha^{-1} \in \Gamma(F^\alpha)$,*

(7) *for F, $G \in E^*$, if F and G are comparable in E^* and $\alpha \in \Gamma(F) \cap \Gamma(G)$ and if there exist elements $f \in F$ and $g \in G$ such that $f \leqslant g$, then there exist elements $h \in F^\alpha$ and $k \in G^\alpha$ such that $h \leqslant k$.*

Proof. (1) If $D(F)$ is a $\mathscr{D}$-class of L-type, then, by Lemmas 4.4 and 3.3, all $\mathscr{D}_E$-classes in $\Delta(F)$ are of L-type. If $D(F)$ is of R-type, then all $\mathscr{D}_E$-classes in $\Delta(F)$ are of R-type.

(2) Let $\alpha \in \Gamma$. Then there exists an element $a \in S$ such that $a\sigma^\natural = \alpha$. Taking an inverse a' of a we have, by definition, $\alpha \in \Gamma(aa')$ and so $\alpha \in \Gamma(D_E(aa'))$ with $D_E(aa') \in E^*$.

(3) Clearly, for $f \in F$ we have $f\mathscr{R}f$ and $f\sigma^{\natural} = 1$. Hence $1 \in \Gamma(F)$. Moreover, f itself is an inverse of f and so, by definition, $F^1 = F$.

(4) Let $f \in F$. Since $\alpha \in \Gamma(F)$ there exists an element $a \in S$ such that $f\mathscr{R}a$ and $a\sigma^{\natural} = \alpha$. Taking an inverse a' of a we have $aa'\mathscr{R}a\mathscr{R}f$ and so $aa' \in F$. Hence, by Lemma 4.5, we have $\alpha \in \Gamma(G)$ and, for $g \in G$, $a'ga \in G^{\alpha}$. But, by definition, $a'a \in F^{\alpha}$ and $a'ga = (a'a)(a'ga)$. Hence $G^{\alpha} = F^{\alpha} \circ G^{\alpha}$ and so $G^{\alpha} \leqslant F^{\alpha}$.

(5) Since $\alpha \in \Gamma(F)$ there exists, for $f \in F$, an element $a \in S$ with $a\mathscr{R}f$ and $a\sigma^{\natural} = \alpha$. Hence, taking an inverse a' of a, we have $a'a \in F^{\alpha}$. Since $\beta \in \Gamma(F^{\alpha})$, there exists an element $b \in S$ with $a'a\mathscr{R}b$ and $b\sigma^{\natural} = \beta$. Hence $f\mathscr{R}a = aa'a\mathscr{R}ab$ and $(ab)\sigma^{\natural} = (a\sigma^{\natural})(b\sigma^{\natural}) = \alpha\beta$ and so $\alpha\beta \in \Gamma(F)$. We take an inverse b' of b. Then, by Lemma 1.9, $b'a'$ is an inverse of ab and so, by definition, $b'a'ab \in F^{\alpha\beta}$. On the other hand, $a'a\mathscr{R}a'ab$, $(a'ab)\sigma^{\natural} = ((a'a)\sigma^{\natural})(b\sigma^{\natural}) = b\sigma^{\natural} = \beta$ and, by Lemma 1.9, $b'a'a$ is an inverse of $a'ab$. Hence $b'a'ab = (b'a'a)(a'ab) \in (F^{\alpha})^{\beta}$ and so $F^{\alpha\beta} = (F^{\alpha})^{\beta}$.

(6) We suppose that $f \in F$, $f\mathscr{R}a$, $a\sigma^{\natural} = \alpha$, and that a' is an inverse of a. Then, by definition, $a'a \in F^{\alpha}$. Moreover, $a'a\mathscr{R}a'$ and since $(aa')\sigma^{\natural} = 1$ we have $a'\sigma^{\natural} = (a\sigma^{\natural})^{-1} = \alpha^{-1}$. Hence $\alpha^{-1} \in \Gamma(F^{\alpha})$.

(7) First we suppose that $G \leqslant F$. Since $\alpha \in \Gamma(F)$ there exists an element $a \in S$ such that $a\mathscr{R}f$ and $a\sigma^{\natural} = \alpha$. Taking an inverse a' of a we put $a'fa = h$ and $a'ga = k$. Then $h = a'fa \leqslant a'ga = k$ and, by Lemma 4.5, $h \in F^{\alpha}$ and $k \in G^{\alpha}$. We can prove the assertion similarly in the case where $F \leqslant G$.

For a derived family $\Delta(F)$, if all $\mathscr{D}_E$-classes in $\Delta(F)$ are of L-type, then we say that $\Delta(F)$ is a *derived family of L-type.* If all $\mathscr{D}_E$-classes in $\Delta(F)$ are of R-type, then we say that $\Delta(F)$ is *of R-type.* By (1) of Theorem 4.6, each derived family belongs to one of the two types; that is, L-type and R-type.

LEMMA 4.7. *In an ordered regular proper semigroup S, suppose that $a \in S$, $f \in E$, and $a\mathscr{D}f$. Then a is an element of L-type if and only if $\Delta(D_E(f))$ is a derived family of L-type.*

Proof. First we suppose that a is an element of L-type. Let $G \in \Delta(D_E(f))$. Then, by Lemma 4.4, $G \subseteq D(D_E(f)) = D(f) = D(a)$ and so $D(G) = D(a)$. Hence, by Lemma 3.3, G is a $\mathscr{D}_E$-class of L-type and so $\Delta(D_E(f))$ is of L-type. Next we suppose that $\Delta(D_E(f))$ is of L-type. Let g and h be idempotents of $D(a)$ such that $g\mathscr{R}h$. Then g and h are contained in the same $\mathscr{D}_E$-class, say G. We have $G \subseteq D(a) = D(D_E(f))$ and so, by Lemma 4.4, $G \in \Delta(D_E(f))$. Since $\Delta(D_E(f))$ is of L-type G is a $\mathscr{D}_E$-class of L-type and so $g = h$. Hence $D(a)$ is a $\mathscr{D}$-class of L-type.

LEMMA 4.8. *In an ordered regular proper semigroup S, let $F \in E^*$ and $\alpha \in \Gamma(F)$. Then $\Delta(F) = \Delta(F^{\alpha})$.*

Proof. We suppose that $G \in \Delta(F)$. Then $G = F^{\beta}$ for some $\beta \in \Gamma(F)$. We have $\alpha^{-1} \in \Gamma(F^{\alpha})$, $\beta \in \Gamma(F) = \Gamma(F^{\alpha\alpha^{-1}})$ and so $\alpha^{-1}\beta \in \Gamma(F^{\alpha})$ and $F^{\alpha\alpha^{-1}\beta} = F^{\beta} = G$. Hence $G \in \Delta(F^{\alpha})$ and so $\Delta(F) \subseteq \Delta(F^{\alpha})$. Replacing F by F^{α} and α by α^{-1} we have $\Delta(F^{\alpha}) \subseteq \Delta(F)$.

For $a \in S$ we associate a triple (α, A_1, A_2), where

$$\alpha = a\sigma^{\natural},$$

$$A_1 = \{aa';\ a' \text{ is an inverse of } a\},$$

$$A_2 = \{a'a;\ a' \text{ is an invers of } a\}.$$

Then $aa'\mathscr{R}a$, $a'a\mathscr{L}a$ and so all elements of A_1 are $\mathscr{R}$-equivalent to one another and all elements of A_2 are $\mathscr{L}$-equivalent to one another. Also, we have $\alpha \in \Gamma(D_E(A_1))$ and $(D_E(A_1))^{\alpha} = D_E(A_2)$. Now we suppose that a is an element of L-type. Then, since $A_1 \subseteq D(a)$, A_1 is a one-element subset of E. Let $e \in D_E(A_2)$. Then $a\mathscr{L}e\mathscr{L}a'a$ and so $ae = a$ and $ea'a = e$. By Lemma 1.9, ea' is an inverse of $ae = a$ and so $e = ea'a \in A_2$. Therefore $A_2 = D_E(A_2)$ and so A_2 is a $\mathscr{D}_E$-class. Moreover, by Lemma 4.7, $\Delta(D_E(A_1))$ is a derived family of L-type. Thus α, A_1, and A_2 satisfy the conditions that A_1 is a one-element subset of E, $\alpha \in \Gamma(D_E(A_1))$, $A_2 = (D_E(A_1))^{\alpha}$, and $\Delta(D_E(A_1))$ is a derived family of L-type. Conversely, let (α, A_1, A_2) be a triple satisfying these conditions. Then, putting $A_1 = \{a_1\}$, there exists an element $a \in S$ such that $\alpha = a\sigma^{\natural}$ and $a\mathscr{R}a_1$. By Lemma 4.7, a is an element of L-type. Moreover, taking an inverse a' of a, we have $aa'\mathscr{R}a\mathscr{R}a_1$ and so

$$\{aa';\ a' \textit{ is an inverse of } a\} = \{a_1\} = A_1 .$$

Also

$$\{a'a;\ a' \textit{ is an inverse of } a\} = (D_E(\{a_1\}))^{\alpha} = (D_E(A_1))^{\alpha} = A_2 .$$

Thus there exists an element a of L-type such that $\alpha = a\sigma^{\natural}$,

$$A_1 = \{aa';\ a' \textit{ is an inverse of } a\}, \qquad \text{and} \qquad A_2 = \{a'a;\ a' \textit{ is an inverse of } a\}.$$

Let a^* be another such element. Then aRa_1Ra^* and $a\sigma^{\natural} = \alpha = \alpha^*\sigma^{\natural}$. Hence, by Lemma 4.1, we have $a = a^*$. Therefore the element a is determined uniquely by α, A_1, and A_2. Similarly, we can prove that if a is not an element of L-type, then α, A_1 and A_2 (which satisfy the original conditions) satisfy the conditions that A_1 is a $\mathscr{D}_E$-class, $\alpha \in \Gamma(A_1)$, A_2 is a one-element subset of E, $(D_E(A_2)) = (D_E(A_1))^{\alpha}$, and $\Delta(D_E(A_1))$ is not a derived family of L-type. Conversely, α, A_1, A_2 satisfying these conditions determine uniquely an element a which is not of L-type and which satisfies the original conditions.

For $a \in S$ the triple (α, A_1, A_2) associated with a is called the *representation* of a. The components α, A_1, and A_2 are called the *Γ-component*, *R-component*, and *L-component* of a, respectively. In the following, when a has the representation (α, A_1, A_2) we usually identify the element and its representation and write $a = (\alpha, A_1, A_2)$.

The following definition and lemma are related to the set E of idempotents of an ordered regular proper semigroup S or, more generally, an ordered idempotent semigroup E. For a $\mathscr{D}_E$-class F and an element $e \in E$, we define $(F)_{e+}$ and $(F)_{e-}$ by

$$(F)_{e+} = \min \{x; e \leqslant x \text{ and } x \in F\},$$

$$(F)_{e-} = \max \{x; x \leqslant e \text{ and } x \in F\}.$$

LEMMA 4.9. *In an ordered idempotent semigroup E, let F be a $\mathscr{D}_E$-class and let $e \in E$ such that $F \leqslant D_E(e)$. If there is an element $f \in F$ such that $e \leqslant f$, then $(F)_{e+}$ exists and $(F)_{e+} = efe$. If there is an element $f' \in F$ such that $f' \leqslant e$, then $(F)_{e-}$ exists and $(F)_{e-} = ef'e$.*

Proof. First we suppose that there is $f \in F$ such that $e \leqslant f$. Then, by Lemma 1.5, if F is a $\mathscr{D}_E$-class of L-type, then $(F)_{e+}$ exists and $(F)_{e+} = ef$. If F is of R-type, then $(F)_{e+}$ exists and $(F)_{e+} = fe$. But $ef \in F, fe \in F$ and so, if F is of L-type, then $efe = (ef)(fe) = ef$. If F is of R-type, then $efe = fe$. Thus, in both cases, $(F)_{e+}$ exists and $(F)_{e+} = efe$. The second assertion can be proved similarly.

THEOREM 4.10. *In an ordered regular proper semigroup S, let $a = (\alpha, A_1, A_2), b = (\beta, B_1, B_2), ab = (\gamma, C_1, C_2)$, and $F = D_E(A_2) \circ D_E(B_1)$. Then $\alpha\beta = \gamma$, $\alpha^{-1} \in \Gamma(F)$, $\beta \in \Gamma(F)$, $D_E(C_1) = F^{\alpha^{-1}}$, and $D_E(C_2) = F^{\beta}$. Moreover,*

(1) *if $\Delta(F)$ is a derived family of L-type and if, for each $a_2 \in A_2$, there exists $b_1 \in B_1$ such that $a_2 \leqslant b_1$, then*

$$C_1 = \{(F^{\alpha^{-1}})_{a_1+}\} \quad \text{for every} \quad a_1 \in A_1 \quad \text{and} \quad C_2 = F^{\beta};$$

(2) *if $\Delta(F)$ is of L-type and if there exists $a_2 \in A_2$ such that $b_1 < a_2$ for every $b_1 \in B_1$, then*

$$C_1 = \{(F^{\alpha^{-1}})_{a_1-}\} \quad \text{for every} \quad a_1 \in A_1 \quad \text{and} \quad C_2 = F^{\beta};$$

(3) *if $\Delta(F)$ is not of L-type and if, for each $b_1 \in B_1$, there exists $a_2 \in A_2$ such that $b_1 \leqslant a_2$, then*

$$C_1 = F^{\alpha^{-1}} \quad \text{and} \quad C_2 = \{(F^{\beta})_{b_2+}\} \quad \text{for every} \quad b_2 \in B_2;$$

(4) *if $\Delta(F)$ is not of L-type and if there exists $b_1 \in B_1$ such that $a_2 < b_1$ for every $a_2 \in A_2$, then*

$$C_1 = F^{\alpha^{-1}} \quad \textit{and} \quad C_2 = \{(F^{\beta})_{b_2-}\} \quad \textit{for every} \quad b_2 \in B_2 .$$

Proof. We have $\gamma = (ab)\,\sigma^{\natural} = (a\sigma^{\natural})\,(b\sigma^{\natural}) = \alpha\beta$. Since $\alpha \in \Gamma(D_E(A_1))$, $D_E(A_2) = (D_E(A_1))^{\alpha}$, and $F = D_E(A_2) \circ D_E(B_1) \leqslant D_E(A_2)$, we have, by Theorem 4.6, $\alpha^{-1} \in \Gamma(F)$. Similarly, since $\beta \in \Gamma(D_E(B_1))$ and $F \leqslant D_E(B_1)$ we have $\beta \in \Gamma(F)$. Let a' and b' be inverses of a and b, respectively. Then $a'a \in A_2 \subseteq D_E(A_2)$, $bb' \in B_1 \subseteq D_E(B_1)$, and so $a'abb' \in D_E(A_2) \circ D_E(B_1) = F$. Now $F \leqslant D_E(A_2)$, $a'\sigma^{\natural} = (a\sigma^{\natural})^{-1} = \alpha^{-1}$, and a is an inverse of a'. Hence, by Lemma 4.5, $abb'a' = a(a'abb')\,a' \in F^{\alpha^{-1}}$. On the other hand, by Lemma 1.9, $b'a'$ is an inverse of ab, and so $abb'a' \in C_1 \subseteq D_E(C_1)$. Hence $F^{\alpha^{-1}} = D_E(C_1)$. The assertion that $D_E(C_2) = F^{\beta}$ can be proved in a similar way. Next we proceed to the proof of the assertion (1) and suppose that $\Delta(F)$ is of L-type and, for each $a_2 \in A_2$, there exists $b_1 \in B_1$ such that $a_2 \leqslant b_1$. Then, by Lemma 4.8, $\Delta(D_E(C_1)) = \Delta(F^{\alpha^{-1}}) = \Delta(F)$ and, since $\Delta(F)$ is of L-type, C_1 is a one-element set and C_2 is a $\mathscr{D}_E$-class. Hence $C_2 = D_E(C_2) = F^{\beta}$. We take $a_1 \in A_1$. Then $a_1 = aa'$ for some inverse a' of a. Also, we take an inverse b' of b. Since $a'a \in A_2$ we have, by assumption, $a'a \leqslant g$ for some $g \in B_1$. Hence, by Lemma 1.1, $a'a \leqslant a'ag$. But, since $g \in B_1$ and $bb' \in B_1$ we have $g\mathscr{R}bb'$ and so $a'ag\mathscr{R}a'abb'$. Moreover, $a'ag \in F$ and $a'abb' \in F$. By assumption $\Delta(F)$ is a derived family of L-type and so F is a $\mathscr{D}_E$-class of L-type. Hence we have $a'ag = a'abb'$ and so $aa' = a(a'a)\,a' \leqslant a(a'ag)\,a' = a(a'abb')\,a' = abb'a'$ and $abb'a' \in F^{\alpha^{-1}}$. Hence, by Lemma 4.9, $(F^{\alpha^{-1}})_{aa'+}$ exists and $(F^{\alpha^{-1}})_{aa'+} = aa'(abb'a')\,aa' = abb'a'$. Thus we have

$$C_1 = \{abb'a'\} = \{(F^{\alpha^{-1}})_{a_1+}\}.$$

Now we proceed to the proof of the assertion (2) and suppose that $\Delta(F)$ is of L-type and there is an element a_2 of A_2 such that $b_1 < a_2$ for every $b_1 \in B_1$. Taking a' and b' as above we have $bb' < a_2$. Since $a_2 \in A_2$ and $a'a \in A_2$ we have $a_2\mathscr{L}a'a$ and so $a'a = a'aa_2 \geqslant a'abb'$. Hence

$$aa' = a(a'a)\,a' \geqslant a(a'abb')\,a' = abb'a'.$$

Now we can proceed in a similar way as above and prove the assertion (2). The assertions (3) and (4) can be proved similarly.

Lemma 4.11. *In an ordered regular proper semigroup S, let F and G be two $\mathscr{D}_E$-classes such that $\alpha \in \Gamma(F) \cap \Gamma(G)$. Then $\alpha \in \Gamma(F \circ G)$ and $(F \circ G)^{\alpha} = F^{\alpha} \circ G^{\alpha}$.*

Proof. We apply Theorem 4.6 repeatedly. Since $F \circ G \preccurlyeq F$ and $\alpha \in \Gamma(F)$ we have $\alpha \in \Gamma(F \circ G)$. Moreover, $(F \circ G)^\alpha \preccurlyeq F^\alpha$, $(F \circ G)^\alpha \preccurlyeq G^\alpha$, and so $(F \circ G)^\alpha \preccurlyeq F^\alpha \circ G^\alpha$. On the other hand $\alpha^{-1} \in \Gamma(F^\alpha) \cap \Gamma(G^\alpha)$ and so, similarly as above, we obtain $(F^\alpha \circ G^\alpha)^{\alpha^{-1}} \preccurlyeq F^{\alpha\alpha^{-1}} \circ G^{\alpha\alpha^{-1}} = F \circ G$. Hence $F^\alpha \circ G^\alpha = (F^\alpha \circ G^\alpha)^{\alpha^{-1}\alpha} \preccurlyeq (F \circ G)^\alpha$ and so $F^\alpha \circ G^\alpha = (F \circ G)^\alpha$.

THEOREM 4.12. *In an ordered regular proper semigroup S we have $a = (\alpha, A_1, A_2) \leqslant b = (\beta, B_1, B_2)$ if and only if either*

(1) $\alpha < \beta$, *or*

(2) $\alpha = \beta$, *$\Delta(D_E(A_1) \circ D_E(B_1))$ is a derived family of L-type and there exist elements $f \in A_1$ and $g \in B_1$ such that $f \leqslant g$, or*

(3) $\alpha = \beta$, *$\Delta(D_E(A_1) \circ D_E(B_1))$ is not of L-type and there exist elements $h \in A_2$ and $k \in B_2$ such that $h \leqslant k$.*

Proof. First we suppose that $\alpha < \beta$. Then, by the definition of the order of Γ, we have $a < b$. Next we suppose that condition (2) holds. First we consider the case when $D_E(A_1) \preccurlyeq D_E(B_1)$. We take inverses a' and b' of a and b, respectively. Since $\Delta(D_E(A_1) \circ D_E(B_1)) = \Delta(D_E(A_1))$ is of L-type we have $A_1 = \{aa'\}$ and so, by assumption, there exists $g \in B_1$ such that $aa' \leqslant g$. Now we suppose that $bb' < aa'$ is true. Then $bb' < aa' \leqslant g$ and so, by Lemma 1.3, $D_E(A_1) = D_E(aa') \succcurlyeq D_E(bb') \circ D_E(g) = D_E(B_1)$. Hence $D_E(A_1) = D_E(B_1)$ and so $\Delta(D_E(B_1))$ is of L-type. Hence B_1 is a one-element set which contradicts that B_1 contains two distinct elements bb' and g. Thus we have $aa' \leqslant bb'$. Now $D_E(A_1) \in \Delta(D_E(A_1))$ and so $D_E(A_1)$ is a D_E-class of L-type. Moreover, $aa'bb' \in D_E(A_1)$ and $aa' \in D_E(A_1)$ and so $aa'bb' = aa'(aa'bb') = aa'$. Hence $aa'b\mathscr{R}aa'\mathscr{R}a$. Moreover,

$$(aa'b)\,\sigma^\natural = (aa')\,\sigma^\natural b\sigma^\natural = b\sigma^\natural = \beta = \alpha = a\sigma^\natural.$$

Hence, by Lemma 4.1, $aa'b = a$ and so $a = aa'b \leqslant bb'b = b$. In the case where $D_E(A_1) \succcurlyeq D_E(B_1)$ we can prove that $a \leqslant b$ in a similar way. Now we consider the general case. By assumption there exist elements $f \in A_1$ and $g \in B_1$ such that $f \leqslant g$. By Lemma 1.1, $f \leqslant fg \leqslant g$ and $fg \in D_E(A_1) \circ D_E(B_1)$. Since $\Delta(D_E(A_1) \circ D_E(B_1))$ is of L-type $(\alpha, \{fg\}, (D_E(A_1) \circ D_E(B_1))^\alpha)$ is the representation of an element c. Since $D_E(A_1) \circ D_E(B_1) \preccurlyeq D_E(A_1)$ and $D_E(A_1) \circ D_E(B_1) \preccurlyeq D_E(B_1)$ we can apply the results obtained above and then $a \leqslant c \leqslant b$. Finally we suppose that condition (3) holds. By Lemmas 4.11 and 4.8

$$\Delta(D_E(A_2) \circ D_E(B_2)) = \Delta((D_E(A_1))^\alpha \circ (D_E(B_1))^\alpha) = \Delta((D_E(A_1) \circ D_E(B_1))^\alpha)$$
$$= \Delta(D_E(A_1) \circ D_E(B_1)).$$

Hence $\Delta(D_E(A_2) \circ D_E(B_2))$ is not of L-type and we can prove that $a \leqslant b$ in a similar way.

Now we proceed to the proof of the converse part and suppose that $a \leqslant b$. If $\alpha > \beta$ were true, then, by the fact just proved, we would have $a > b$. Thus $\alpha \leqslant \beta$. Now we suppose that $\alpha = \beta$ and $\Delta(D_E(A_1) \circ D_E(B_1))$ is of L-type. We take $f \in A_1$ and $g \in B_1$ arbitrarily. If $f > g$, then, by the fact just proved, $a \geqslant b$ and so $a = b$. Hence $A_1 = B_1$ and so trivially we have $f = g'$ for $f \in A_1$ and $g' \in B_1$. Thus in this case (2) holds. Similarly, if $\alpha = \beta$ and $\Delta(D_E(A_1) \circ D_E(B_1))$ is not of L-type, then we have (3).

5. The Characterization Theory

In this section we argue conversely and prove that the theorems in Section 4 really characterize ordered regular proper semigroups.

Theorem 5.1. *Let E' be an ordered idempotent semigroup and let Γ' be a simply-ordered group with the identity* 1. *Suppose that for each element F of the associated semilattice E'^* of E', $\Gamma(F)$ is defined to be a subset of Γ' and, for each $F \in E'^*$ and $\alpha \in \Gamma(F)$, F^α is defined to be an element of E'^*. We denote $\Delta(F) = \{F^\alpha; \alpha \in \Gamma(F)\}$ and suppose that the following conditions are satisfied:*

(i) *for each $F \in E'^*$, either all $\mathcal{D}_{E'}$-classes in $\Delta(F)$ are of L-type or all $\mathcal{D}_{E'}$-classes in $\Delta(F)$ are of R-type;*

(ii) $\bigcup_{F \in E'^*} \Gamma(F) = \Gamma'$;

(iii) *for every $F \in E'^*$, we have $1 \in \Gamma(F)$ and $F^1 = F$;*

(iv) *for $F, G \in E'^*$, if $G \leqslant F$ and $\alpha \in \Gamma(F)$, then $\alpha \in \Gamma(G)$ and $G^\alpha \leqslant F^\alpha$;*

(v) *for $F \in E'^*$, if $\alpha \in \Gamma(F)$ and $\beta \in \Gamma(F^\alpha)$, then $\alpha\beta \in \Gamma(F)$ and $F^{\alpha\beta} = (F^\alpha)^\beta$;*

(vi) *for $F \in E'^*$, if $\alpha \in \Gamma(F)$, then $\alpha^{-1} \in \Gamma(F^\alpha)$,*

(vii) *for $F, G \in E'^*$, if F and G are comparable in E'^* and $\alpha \in \Gamma(F) \cap \Gamma(G)$ and if there exist elements $f \in F$ and $g \in G$ such that $f \leqslant g$, then there exist elements $h \in F^\alpha$ and $k \in G^\alpha$ such that $h \leqslant k$.*

If all $\mathcal{D}_{E'}$-classes in $\Delta(F)$ are of L-type, then $\Delta(F)$ is of L-type. We denote by S' the set of all those triples (α, A_1, A_2) which satisfy either:

(viii) *$\Delta(D_{E'}(A_1))$ is of L-type, A_1 is a one-element subset of E', $\alpha \in \Gamma(D_{E'}(A_1))$ and $A_2 = (D_{E'}(A_1))^\alpha$, or*

(ix) *$\Delta(D_{E'}(A_1))$ is not of L-type, A_1 is a $\mathcal{D}_{E'}$-class, $\alpha \in \Gamma(A_1)$; A_2 is a one-element subset of E', and $D_{E'}(A_2) = A_1{}^\alpha$.*

We define the product in S' by

$$(\alpha, A_1, A_2)(\beta, B_1, B_2) = (\gamma, C_1, C_2),$$

where, putting $F = D_{E'}(A_2) \circ D_{E'}(B_1)$,

(x) *if* $\Delta(F)$ *is of L-type and if, for each element* a_2 *of* A_2, *there exists an element* b_1 *of* B_1 *such that* $a_2 \leqslant b_1$, *then*

$$\gamma = \alpha\beta, \qquad C_1 = \{(F^{\alpha^{-1}})_{f+}\}, \qquad C_2 = F^{\beta} \qquad \textit{where} \qquad f \in A_1;$$

(xi) *if* $\Delta(F)$ *is of L-type and if there is an element* a_2 *of* A_2 *such that* $b_1 < a_2$ *for every* $b_1 \in B_1$, *then*

$$\gamma = \alpha\beta, \qquad C_1 = \{(F^{\alpha^{-1}})_{f-}\}, \qquad C_2 = F^{\beta} \qquad \textit{where} \qquad f \in A_1;$$

(xii) *if* $\Delta(F)$ *is not of L-type and if, for each element* b_1 *of* B_1, *there exists an element* a_2 *of* A_2 *such that* $b_1 \leqslant a_2$, *then*

$$\gamma = \alpha\beta, \qquad C_1 = F^{\alpha^{-1}}, \qquad C_2 = \{(F^{\beta})_{k+}\} \qquad \textit{where} \qquad k \in B_2;$$

(xiii) *if* $\Delta(F)$ *is not of L-type and if there is an element* b_1 *of* B_1 *such that* $a_2 < b_1$ *for every* $a_2 \in A_2$, *then*

$$\gamma = \alpha\beta, \qquad C_1 = F^{\alpha^{-1}}, \qquad C_2 = \{(F^{\beta})_{k-}\} \qquad \textit{where} \qquad k \in B_2.$$

Also we define the order in S' *by*

$$(\alpha, A_1, A_2) \leqslant (\beta, B_1, B_2) \qquad \textit{if and only if either}$$

(xiv) $\alpha < \beta$, *or*

(xv) $\alpha = \beta$, $\Delta(D_{E'}(A_1) \circ D_{E'}(B_1))$ *is of L-type and there exist* $f \in A_1$ *and* $g \in B_1$ *such that* $f \leqslant g$, *or*

(xvi) $\alpha = \beta$, $\Delta(D_{E'}(A_1) \circ D_{E'}(B_1))$ *is not of L-type and there exist* $h \in A_2$ *and* $k \in B_2$ *such that* $h \leqslant k$.

Then S' *is an ordered regular proper semigroup.*

Proof. The proof of this theorem is divided into several parts. For brevity, for the theorems in this section, we often give only the proofs of one case. Also, we use $\mathscr{D}$-classes in E' exclusively and we denote $D(A)$ in place of $D_{E'}(A)$.

(1) *If* $F \in E'^*$ *and* $\alpha \in \Gamma(F)$, *then* $\Delta(F) = \Delta(F^{\alpha})$.

For the proof, see the proof of Lemma 4.8.

(2) *If* (α, A_1, A_2), $(\beta, B_1, B_2) \in S'$, *then* $(\alpha, A_1, A_2)(\beta, B_1, B_2)$ *is well-defined and belongs to* S'.

We give the proof only in the case (x). We have $F \leqslant D(A_2)$, $F \leqslant D(B_1)$, $\alpha \in \Gamma(D(A_1))$, $D(A_2) = (D(A_1))^{\alpha}$, and $\beta \in \Gamma(D(B_1))$. Hence, by (vi),

$\alpha^{-1} \in \Gamma(D(A_2))$ and so, by (iv), $\alpha^{-1} \in \Gamma(F)$ and $\beta \in \Gamma(F)$. Therefore $F^{\alpha^{-1}}$ and F^β are definable. By assumption there exist $a_2 \in A_2$ and $b_1 \in B_1$ with $a_2 \leqslant b_1$. Then, by Lemma 1.1, $a_2 \leqslant a_2b_1 \leqslant b_1$ and $a_2b_1 \in D(A_2) \circ D(B_1) = F$. Hence, by (vii), there exist $f' \in (D(A_2))^{\alpha^{-1}}$ and $t \in F^{\alpha^{-1}}$ such that $f' \leqslant t$. But, by (v) and (iii), $(D(A_2))^{\alpha^{-1}} = D(A_1)$ and so $f' \in D(A_1)$. Moreover, by (iv), $D(A_1) = (D(A_2))^{\alpha^{-1}} \succcurlyeq F^{\alpha^{-1}}$. Since $F^{\alpha^{-1}} \in \Delta(F)$, $F^{\alpha^{-1}}$ is of L-type. Hence, by Lemma 1.5, $(F^{\alpha^{-1}})_{f'+}$ exists and $f't = (F^{\alpha^{-1}})_{f'+}$. Now we suppose that $F^{\alpha^{-1}} \neq D(A_1)$ and assume that there is $f \in A_1$ such that either $(F^{\alpha^{-1}})_{f+}$ does not exist or $(F^{\alpha^{-1}})_{f+}$ exists but is not equal to $f't$. Then either $f't < f$ or there exists $t' \in F^{\alpha^{-1}}$ such that $f \leqslant t' < f't$. In the latter case we have $t' < f'$ since $f't = (F^{\alpha^{-1}})_{f'+}$. Thus, in both cases there exists an element of $F^{\alpha^{-1}}$ which lies between f and f'. Hence, by Lemma 1.3, $F^{\alpha^{-1}} \succcurlyeq D(f) \circ D(f') = D(A_1)$ and so $F^{\alpha^{-1}} = D(A_1)$, which is a contradiction. Hence, if $F^{\alpha^{-1}} \neq D(A_1)$ then, for every $f \in A_1$, $(F^{\alpha^{-1}})_{f+}$ exists and is equal to one another. If $F^{\alpha^{-1}} = D(A_1)$, then, by (1) above $\Delta(D(A_1)) = \Delta(F^{\alpha^{-1}}) = \Delta(F)$ is of L-type and so A_1 is a one-element set $\{f\}$ and $(D(A_1))_{f+} = f$. Hence we obtain the same conclusion. Thus C_1 is well-defined irrespective of the choice of an element $f \in A_1$. Moreover, $D(C_1) = F^{\alpha^{-1}}$ and so, by (1), $\Delta(D(C_1)) = \Delta(F^{\alpha^{-1}}) = \Delta(F)$. Hence $\Delta(D(C_1))$ is of L-type and, since $\alpha \in \Gamma(F^{\alpha^{-1}})$ and $\beta \in \Gamma(F) = \Gamma(F^{\alpha^{-1}\alpha})$, by (v) $\alpha\beta \in \Gamma(F^{\alpha^{-1}}) = \Gamma(D(C_1))$ and $C_2 = F^\beta = F^{\alpha^{-1}\alpha\beta} = (D(C_1))^{\alpha\beta}$. Hence $(\gamma, C_1, C_2) \in S'$.

(3) *If* $F, G \in E'^*$ *and* $\alpha \in \Gamma(F) \cap \Gamma(G)$, *then* $\alpha \in \Gamma(F \circ G)$ *and* $(F \circ G)^\alpha = F^\alpha \circ G^\alpha$.

For the proof, see the proof of Lemma 4.11.

(4) *If* $(\alpha, A_1, A_2), (\beta, B_1, B_2), (\gamma, C_1, C_2) \in S'$, *then*

$$((D(A_2) \circ D(B_1))^\beta \circ D(C_1))^{\beta^{-1}\alpha^{-1}} = (D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}})^{\alpha^{-1}}$$

and these are equal to either $(D(A_2) \circ D(B_1))^{\alpha^{-1}}$ *or* $(D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$.

In fact, $D(A_2) \circ D(B_1) \preccurlyeq D(B_1)$ and $\beta \in \Gamma(D(B_1))$. Hence $\beta \in \Gamma(D(A_2) \circ D(B_1))$ and $(D(A_2) \circ D(B_1))^\beta \preccurlyeq (D(B_1))^\beta = D(B_2)$. Therefore

$$(D(A_2) \circ D(B_1))^\beta \circ D(C_1) = (D(A_2) \circ D(B_1))^\beta \circ D(B_2) \circ D(C_1).$$

Since $\beta^{-1} \in \Gamma((D(B_1))^\beta) = \Gamma(D(B_2))$ and $D(B_2) \circ D(C_1) \preccurlyeq D(B_2)$ we have $\beta^{-1} \in \Gamma(D(B_2) \circ D(C_1))$. Moreover, $\beta^{-1} \in \Gamma((D(A_2) \circ D(B_1))^\beta)$ and so, by (3),

$$\begin{aligned}((D(A_2) \circ D(B_1))^\beta \circ D(C_1))^{\beta^{-1}} &= (D(A_2) \circ D(B_1))^{\beta\beta^{-1}} \circ (D(B_2) \circ D(C_1))^{\beta^{-1}} \\ &= (D(A_2) \circ D(B_1)) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}}.\end{aligned}$$

But

$$(D(B_2) \circ D(C_1))^{\beta^{-1}} \leqslant (D(B_2))^{\beta^{-1}} = (D(B_1))^{\beta\beta^{-1}} = D(B_1).$$

Hence

$$((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}} = D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}}.$$

Since

$$\alpha^{-1} \in \Gamma(D(A_2)) \qquad \text{and} \qquad D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}} \leqslant D(A_2)$$

we have

$$\alpha^{-1} \in \Gamma(D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}}) = \Gamma(((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}})$$

and so

$$((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}\alpha^{-1}} = (D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}})^{\alpha^{-1}}.$$

Moreover,

$$D(A_2) \circ D(B_1) \leqslant D(B_1), \qquad (D(B_2) \circ D(C_1))^{\beta^{-1}} \leqslant (D(B_2))^{\beta^{-1}} = D(B_1)$$

and so, by Lemma 1.6, $D(A_2) \circ D(B_1)$ and $(D(B_2) \circ D(C_1))^{\beta^{-1}}$ are comparable in E'^*. Hence

$$\begin{aligned} &((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}\alpha^{-1}} \\ &= ((D(A_2) \circ D(B_1)) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}})^{\alpha^{-1}} \end{aligned}$$

is either $(D(A_2) \circ D(B_1))^{\alpha^{-1}}$ or $(D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$.

(5) *S' is a semigroup, that is, if $(\alpha, A_1, A_2), (\beta, B_1, B_2), (\gamma, C_1, C_2) \in S'$, then*

$$((\alpha, A_1, A_2)(\beta, B_1, B_2))(\gamma, C_1, C_2) = (\alpha, A_1, A_2)((\beta, B_1, B_2)(\gamma, C_1, C_2)).$$

In fact, we put, by (4),

$$P = ((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}\alpha^{-1}} = (D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}})^{\alpha^{-1}}.$$

Also, we put

$$\begin{aligned} (\alpha, A_1, A_2)(\beta, B_1, B_2) &= (\delta, D_1, D_2), \\ (\beta, B_1, B_2)(\gamma, C_1, C_2) &= (\kappa, K_1, K_2), \\ ((\alpha, A_1, A_2)(\beta, B_1, B_2))(\gamma, C_1, C_2) &= (\lambda, L_1, L_2), \\ (\alpha, A_1, A_2)((\beta, B_1, B_2)(\gamma, C_1, C_2)) &= (\mu, M_1, M_2). \end{aligned}$$

Then $\delta = \alpha\beta$, $\kappa = \beta\gamma$ and so $\lambda = \alpha\beta\gamma = \mu$. Moreover,

$$D(L_1) = (D(D_2) \circ D(C_1))^{\delta^{-1}} = ((D(A_2) \circ D(B_1))^{\beta} \circ D(C_1))^{\beta^{-1}\alpha^{-1}} = P,$$

$$D(M_1) = (D(A_2) \circ D(K_1))^{\alpha^{-1}} = (D(A_2) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}})^{\alpha^{-1}} = P.$$

Also, by (4), P is equal to either $(D(A_2) \circ D(B_1))^{\alpha^{-1}}$ or $(D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$. In this paper we prove $(\lambda, L_1, L_2) = (\mu, M_1, M_2)$ only in the case when $\Delta(P)$ is of L-type, $P = (D(A_2) \circ D(B_1))^{\alpha^{-1}}$, and $P \neq (D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$. We have $D(D_1) = (D(A_2) \circ D(B_1))^{\alpha^{-1}} = P = D(L_1) = D(M_1)$ and so, since $\Delta(P)$ is of L-type, D_1, L_1 and M_1 are all one-element sets. We put $D_1 = \{d_1\}, L_1 = \{l_1\}$ and $M_1 = \{m_1\}$. Since $(\delta, D_1, D_2)(\gamma, C_1, C_2) = (\lambda, L_1, L_2)$ and $\Delta(P)$ is of L-type, we have either $l_1 = (P)_{d_1+}$ or $l_1 = (P)_{d_1-}$. But $d_1 \in D(D_1) = P$ and so, in both cases, we have $l_1 = d_1$. If, for each $a_2 \in A_2$, there exists an element $b_1 \in B_1$ such that $a_2 \leqslant b_1$, then $a_2 \leqslant a_2 b_1 \leqslant b_1$. Since $P \neq (D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$ we have $P \prec (D(B_2) \circ D(C_1))^{\beta^{-1}\alpha^{-1}}$. Hence, taking $k_1 \in K_1$ arbitrarily,

$$\begin{aligned} D(b_1) \circ D(k_1) &= D(B_1) \circ (D(B_2) \circ D(C_1))^{\beta^{-1}} \\ &= (D(B_2))^{\beta^{-1}} \circ (D(B_2) \circ D(C_1))^{\beta^{-1}} \\ &= (D(B_2) \circ D(C_1))^{\beta^{-1}} \succ P^{\alpha} = D(A_2) \circ D(B_1) = D(a_2 b_1). \end{aligned}$$

Therefore, by Lemma 1.3, $a_2 b_1$ does not lie between b_1 and k_1, and so $a_2 \leqslant a_2 b_1 < k_1$. Hence $l_1 = d_1 = (P)_{a_1+} = m_1$ for $a_1 \in A_1$. In the case when there is an element a_2 of A_2 such that $b_1 < a_2$ for every $b_1 \in B_1$ we can prove $l_1 = d_1 = (P)_{a_1-} = m_1$ in a similar way. Thus in both cases we have $L_1 = \{l_1\} = \{m_1\} = M_1$ and moreover, $L_2 = P^{\alpha\beta\gamma} = M_2$. Hence $(\lambda, L_1, L_2) = (\mu, M_1, M_2)$.

(6) *For $F, G \in E'^*$, if there exist elements $f, f' \in F$ and $g, g' \in G$ such that $f \leqslant g$ and $f' \geqslant g'$, then F and G are comparable in E'^*.*

If $f \leqslant g'$, then we have $f \leqslant g' \leqslant f'$ and so $F = D(f) \circ D(f') \preccurlyeq D(g') = G$. On the other hand, if $g' \leqslant f$, then we have $g' \leqslant f \leqslant g$ and so we obtain $G \preccurlyeq F$ similarly.

(7) *S' is simply-ordered with respect to $\leqslant$.*

In fact, the reflexivity is trivial. Next we prove the antisymmetry and suppose that $(\alpha, A_1, A_2) \leqslant (\beta, B_1, B_2)$ and $(\beta, B_1, B_2) \leqslant (\alpha, A_1, A_2)$. Then we have $\alpha = \beta$. In this paper, we prove $(\alpha, A_1, A_2) = (\beta, B_1, B_2)$ only in the case when $\Delta(D(A_1) \circ D(B_1))$ is of L-type. By definition there exist $f, f' \in A_1$ and $g, g' \in B_1$ such that $f \leqslant g$ and $f' \geqslant g'$. Hence, by (6), two $\mathscr{D}_{E'}$-classes $D(A_1)$ and $D(B_1)$ are comparable in E'^*. Without loss of generality we assume that $D(A_1) \preccurlyeq D(B_1)$. Then $\Delta(D(A_1) \circ D(B_1)) = \Delta(D(A_1))$ is of L-type, and so A_1 is a one-element set. Therefore $f = f'$ and so $g' \leqslant f' = f \leqslant g$. Hence $D(B_1) = D(g) \circ D(g') \preccurlyeq D(f) = D(A_1)$ and so $D(A_1) = D(B_1)$. Therefore $\Delta(D(B_1)) = \Delta(D(A_1))$ is of L-type and so B_1 is also a one-element set. Hence $g = g' = f$ and $A_1 = \{f\} = \{g\} = B_1$. Moreover, $A_2 = (D(A_1))^\alpha = (D(B_1))^\alpha = B_2$. Thus we have $(\alpha, A_1, A_2) = (\beta, B_1, B_2)$. Finally we prove the transitivity and suppose that $(\alpha, A_1, A_2) \leqslant (\beta, B_1, B_2)$ and $(\beta, B_1, B_2) \leqslant (\gamma, C_1, C_2)$. If $\alpha < \beta$ or $\beta < \gamma$, then we have $\alpha < \gamma$ and $(\alpha, A_1, A_2) \leqslant (\gamma, C_1, C_2)$. Now we suppose that $\alpha = \beta = \gamma$. Since $D(A_1) \circ D(B_1) \preccurlyeq D(B_1)$ and $D(B_1) \circ D(C_1) \preccurlyeq D(B_1)$, the $D_{E'}$-classes $D(A_1) \circ D(B_1)$ and $D(B_1) \circ D(C_1)$ are comparable in E'^* by Lemma 1.6. We give the proof only in the case when $\Delta(D(A_1) \circ D(B_1) \circ D(C_1))$ is of L-type and $D(A_1) \circ D(B_1) \preccurlyeq D(B_1) \circ D(C_1)$. Then $\Delta(D(A_1) \circ D(B_1))$ is of L-type and, since $(\alpha, A_1, A_2) \leqslant (\beta, B_1, B_2)$, there exist $f \in A_1$ and $g \in B_1$ such that $f \leqslant g$. Hence, by Lemma 1.1, $f \leqslant fg \leqslant g$. Now we assume it is true that $h < fg$ for every $h \in C_1$. Then, since $h < fg \leqslant g$ we have

$$\begin{aligned} D(A_1) \circ D(B_1) &= D(A_1) \circ D(B_1) \circ D(C_1) \preccurlyeq D(B_1) \circ D(C_1) \\ &= D(g) \circ D(h) \preccurlyeq D(fg) = D(A_1) \circ D(B_1) \end{aligned}$$

and so $D(A_1) \circ D(B_1) = D(B_1) \circ D(C_1)$. Hence $\Delta(D(B_1) \circ D(C_1))$ is of L-type and so, since $(\beta, B_1, B_2) \leqslant (\gamma, C_1, C_2)$ there exist $g' \in B_1$ and $h' \in C_1$ such that $g' \leqslant h'$. Hence $g' \leqslant h' < fg \leqslant g$ and so

$$D(B_1) = D(g') \circ D(g) \preccurlyeq D(h') = D(C_1).$$

Therefore $\Delta(D(B_1)) = \Delta(D(B_1) \circ D(C_1))$ is of L-type and so B_1 is a one-element set, which contradicts that B_1 contains two distinct elements g and g'. This contradiction shows that there exists an element $h \in C_1$ such that $f \leqslant fg \leqslant h$. Hence $D(A_1) \circ D(C_1) = D(f) \circ D(h) \preccurlyeq D(fg) = D(A_1) \circ D(B_1)$ and so $D(A_1) \circ D(C_1) = D(A_1) \circ D(B_1)$. Therefore $\Delta(D(A_1) \circ D(C_1))$ is of L-type and, by (xv), we obtain $(\alpha, A_1, A_2) \leqslant (\gamma, C_1, C_2)$.

(8) *S' is an ordered semigroup.*

By (5) and (7) it suffices to prove the monotone property: if $a \leqslant b$, then $ac \leqslant bc$ and $ca \leqslant cb$. We put $a = (\alpha, A_1, A_2)$, $b = (\beta, B_1, B_2)$, $c = (\gamma, C_1, C_2)$, $ac = (\alpha\gamma, D_1, D_2)$, $bc = (\beta\gamma, F_1, F_2)$, $ca = (\gamma\alpha, G_1, G_2)$, and $cb = (\gamma\beta, H_1, H_2)$. If $\alpha < \beta$, then $\alpha\gamma < \beta\gamma$ and $\gamma\alpha < \gamma\beta$ and so $ac \leqslant bc$ and $ca \leqslant cb$. Next we suppose that $\alpha = \beta$. We have $(D(D_1))^\alpha \preccurlyeq D(C_1)$ and $(D(F_1))^\alpha \preccurlyeq D(C_1)$ and so $(D(D_1))^\alpha$ and $(D(F_1))^\alpha$ are comparable in E'^*, by Lemma 1.6. Hence $D(D_1)$ and $D(F_1)$ are comparable in E'^*. In this paper we prove only $ac \leqslant bc$ in the case when $D(D_1) \preccurlyeq D(F_1)$ and $\Delta(D(D_1))$ is of L-type. Then D_1 is a one-element set, say $D_1 = \{s\}$, and $s = (D(D_1))_{p+}$ or $s = (D(D_1))_{p-}$ for every $p \in A_1$. Now we assume it is true that $t < s$ for every $t \in F_1$. Then we have $t < p$ for every $p \in A_1$; otherwise for some $p \in A_1$ we would have $p \leqslant pt \leqslant t < s$,

$$pt \in D(A_1) \circ (D(B_2) \circ D(C_1))^{\alpha^{-1}} = (D(A_2) \circ D(B_2) \circ D(C_1))^{\alpha^{-1}}$$
$$= D(D_1) \circ D(F_1) = D(D_1)$$

and $s = (D(D_1))_{p+}$, which is a contradiction. We have $q < s$ for every $q \in B_1$; otherwise for some $q \in B_1$ we would have $t < s \leqslant sq \leqslant q$,

$$D(F_1) = (D(B_2) \circ D(C_1))^{\alpha^{-1}} \circ (D(B_2))^{\alpha^{-1}} = (D(B_2) \circ D(C_1))^{\alpha^{-1}} \circ D(B_1)$$
$$= D(t) \circ D(q) \preccurlyeq D(s) = D(D_1).$$

Hence $\Delta(D(F_1)) = \Delta(D(D_1))$ is of L-type and $t = (D(F_1))_{q-}$. On the other hand $sq \in D(D_1) \circ D(B_1) = D(F_1) \circ (D(B_2))^{\alpha^{-1}} = D(F_1)$, which is a contradiction. That $\Delta(D(A_1) \circ D(B_1))$ is of L-type implies $p \leqslant q$ for every $p \in A_1$ and every $q \in B_1$; otherwise there is $p' \in A_1$ and $q' \in B_1$ such that $q' < p'$. Since $(\alpha, A_1, A_2) \leqslant (\beta, B_1, B_2)$, there is $p'' \in A_1$ and $q'' \in B_1$ such that $p'' \leqslant q''$. Hence, by (6), $D(A_1)$ and $D(B_1)$ are comparable in E'^*. If $D(A_1) \preccurlyeq D(B_1)$, then $\Delta(D(A_1)) = \Delta(D(A_1) \circ D(B_1))$ is of L-type and so A_1 is a one-element set. Hence $q' < p' = p'' \leqslant q''$ and so

$$D(B_1) = D(q') \circ D(q'') \preccurlyeq D(p') = D(A_1).$$

Therefore $\Delta(D(B_1)) = \Delta(D(A_1))$ is also of L-type, which contradicts that B_1 has two distinct elements q' and q''. If $D(B_1) \preccurlyeq D(A_1)$ we can obtain a similar contradiction. We have $p < s$ for every $p \in A_1$ and $t < q$ for every $q \in B_1$. Otherwise there is either $p' \in A_1$ such that $q < s \leqslant p'$ for every $q \in B_1$ or $q' \in B_1$ such that $q' \leqslant t < p$ for every $p \in A_1$. Hence either $D(A_1) \circ D(B_1) \preccurlyeq D(D_1)$ or $D(A_1) \circ D(B_1) \preccurlyeq D(F_1)$. In both cases we have $D(A_1) \circ D(B_1) = D(D_1)$ and so $\Delta(D(A_1) \circ D(B_1))$ is of L-type. This contradicts the fact shown above. Thus we have $t < p < s$ and $t < q < s$ for $p \in A_1$ and $q \in B_1$. Hence $t \leqslant pt \leqslant p$ and $pt \in D(F_1) \circ D(A_1) = D(D_1)$. We have $t \leqslant pt \leqslant q$; otherwise we would have $q < pt \leqslant p$ and so $D(A_1) \circ D(B_1) = D(p) \circ D(q) \preccurlyeq D(pt) = D(D_1)$. Hence $\Delta(D(A_1) \circ D(B_1))$ is of L-type, which contradicts the fact shown above. Since $t \leqslant pt \leqslant q$ we have $D(F_1) = D(t) \circ D(q) \preccurlyeq D(pt) = D(D_1)$ and so $\Delta(D(F_1)) = \Delta(D(D_1))$ is of L-type. Hence F_1 is a one-element set $\{t\}$ and, since $p < s$ and $t < q$ we have $s = (D(D_1))_{p+} \neq (D(D_1))_{p-}$ and $t = (D(F_1))_{q-} \neq (D(F_1))_{q+}$. Hence, for $a_2 \in A_2$, there is $c_1 \in C_1$ such that $a_2 \leqslant c_1$, and then $b_2 \in B_2$ such that $c_1 \leqslant b_2$. Hence $D(A_2) \circ D(B_2) \preccurlyeq D(C_1)$ and so $D(A_1) \circ D(B_1) = D(D_1)$. Therefore $\Delta(D(A_1) \circ D(B_1)) = \Delta(D(D_1))$ is of L-type and so $t < p \leqslant q < s$. But then $t < p \leqslant pq \leqslant q < s$, $pq \in D(A_1) \circ D(B_1) = D(D_1)$, which contradicts $s = (D(D_1))_{p+}$. The above consideration shows that $s \leqslant t$ for some $t \in F_1$. Since $s \in D_1$ and $\Delta(D(D_1) \circ D(F_1)) = \Delta(D(D_1))$ is of L-type this shows that $ac \leqslant bc$.

(9) *S' is an ordered regular semigroup.*

By (8), it suffices to prove that S' is regular; that is, for each $a = (\alpha, A_1, A_2) \in S'$, there exists an element $x \in S'$ such that $axa = a$. In this paper we prove this only in the case where $\Delta(D(A_1))$ is of L-type. Then A_1 is a one-element set, say $\{f\}$, $\alpha \in \Gamma(D(A_1))$ and $A_2 = (D(A_1))^{\alpha}$. Taking $g \in A_2$ arbitrarily, we put $x = (\alpha^{-1}, B_1, B_2)$, where $B_1 = \{g\}$ and $B_2 = D(A_1)$. Then $\Delta(D(B_1)) = \Delta(D(A_2)) = \Delta((D(A_1))^{\alpha}) = \Delta(D(A_1))$ is of L-type, $\alpha^{-1} \in \Gamma((D(A_1))^{\alpha}) = \Gamma(D(B_1))$, and $D(B_2) = D(A_1) = (D(A_2))^{\alpha^{-1}} = (D(B_1))^{\alpha^{-1}}$. Hence $x \in S'$. Moreover, $\Delta(D(A_2) \circ D(B_1)) = \Delta(D(B_1))$ is of L-type and $((D(A_2)\, \Delta\, D(B_1))^{\alpha^{-1}})_{f+} = (D(A_1))_{f+} = f = ((D(A_2) \circ D(B_1))^{\alpha^{-1}})_{f-}$. Hence $ax = (1, \{f\}, D(A_1))$. Now we can easily verify that

$$axa = (\alpha, \{f\}, (D(A_1))^{\alpha}) = a.$$

(10) *An element (α, A_1, A_2) of S' is idempotent if and only if $\alpha = 1$.*

In fact, if (α, A_1, A_2) of S' is idempotent, then $\alpha^2 = \alpha$ and so $\alpha = 1$. Conversely, we suppose that $(1, A_1, A_2) \in S'$. In this paper we considered only the case when $\Delta(D(A_1))$ is of L-type. Then A_1 is a one-element set, say $\{f\}$, and $A_2 = D(A_1)$. Now $\Delta(D(A_1) \circ D(A_1)) = \Delta(D(A_1))$ is of L-type and $(D(A_1) \circ D(A_1))_{f+} = (D(A_1) \circ D(A_1))_{f-} = f$. Hence

$$(1, A_1, A_2)^2 = (1, \{f\}, D(A_1)) = (1, A_1, A_2).$$

(11) *S' is an ordered regular proper semigroup.*

By (9), S' is an ordered regular semigroup. Let (α, A_1, A_2) and (β, B_1, B_2) be idempotents of S' and let $(\alpha, A_1, A_2) \leqslant (\gamma, C_1, C_2) \leqslant (\beta, B_1, B_2)$. Then $\alpha \leqslant \gamma \leqslant \beta$ and, by (10), $\alpha = \beta = 1$. Hence $\gamma = 1$ and so, by (10) again, (γ, C_1, C_2) is an idempotent. Therefore, by Theorem 2.11, S' is an ordered regular proper semigroup.

This completes the proof of Theorem 5.1.

Remark. The condition (ii) of Theorem 5.1 is unnecessary for that theorem. But it is essential for Theorem 5.2.

THEOREM 5.2. *In Theorem 5.1, E' is isomorphic as an ordered semigroup with the set E of idempotents of the ordered regular proper semigroup S' and Γ' is isomorphic as an ordered group with the minimum group homomorphic image $\Gamma = S'/\sigma$ of S'. If we identify the corresponding elements in these isomorphisms, then $\Gamma(F)$ and F^{α} in the assumption of Theorem 5.1 coincide with $\Gamma(F)$ and F^{α} defined in Section 4, respectively.*

Proof. We divide the proof into several steps.

(1) *E' is isomorphic as an ordered semigroup with E.*

For $e \in E'$, we define $e\varphi$ as follows:

(a) if $\Delta(D(e))$ is of L-type, then $e\varphi = (1, \{e\}, D(e))$,

(b) if $\Delta(D(e))$ is not of L-type, then $e\varphi = (1, D(e), \{e\})$.

It is easily seen that $e\varphi \in S'$. Moreover, by (10) of the proof of Theorem 5.1, $e\varphi \in E$. Evidently φ is a one-to-one mapping of E' onto E. Now we suppose that $e, f \in E'$ and $e \leqslant f$. Putting $e\varphi = (1, A_1, A_2)$ and $f\varphi = (1, B_1, B_2)$ we have $e \in A_1 \cap A_2$, $f \in B_1 \cap B_2$, and $e \leqslant f$. Hence, by definition, $e\varphi \leqslant f\varphi$. Conversely, we suppose that $e\varphi \leqslant f\varphi$. If $e > f$ were true, then, by the result just proved, $e\varphi \geqslant f\varphi$ and, since φ is one-to-one we have $e\varphi > f\varphi$, which is a contradiction. Hence $e \leqslant f$. Thus E' is order-isomorphic with E. Next we prove that φ is a semigroup homomorphism and let $e, f \in E'$. In this paper we consider only the case where $\Delta(D(e) \circ D(f))$ is of L-type. First we suppose that $D(e) \preccurlyeq D(f)$. Then $\Delta(D(e))$ is of L-type and so $e\varphi = (1, \{e\}, D(e))$. We put $f\varphi = (1, B_1, B_2)$. Then

$$(D(e) \circ D(B_1))_{e+} = (D(e) \circ D(f))_{e+} = (D(e))_{e+} = e = (D(e) \circ D(B_1))_{e-}$$

and $\Delta(D(e) \circ D(B_1)) = \Delta(D(e) \circ D(f))$ is of L-type. On the other hand, $e \in D(e)$, $ef \in D(e)$ and, since $D(e) \in \Delta(D(e))$ the $\mathscr{D}_{E'}$-class $D(e)$ is of L-type. Hence we have $ef = e(ef) = e$. Therefore $(e\varphi)(f\varphi) = (1, \{e\}, D(e)) = e\varphi = (ef)\varphi$. Next we suppose that $D(f) \prec D(e)$. Then we have $e \neq f$ and $f\varphi = (1, \{f\}, D(f))$. We put $e\varphi = (1, A_1, A_2)$. Then, for every $e' \in A_2$, $D(e) \circ D(e') = D(e) \succ D(f)$. Hence f does not lie between e and e'. By Lemma 1.5, if $e < f$, then

$$(D(e) \circ D(f))_{e+} = \min\{y; y \in D(e) \circ D(f) \text{ and } e \leqslant y\} = ef,$$

and if $f < e$, then

$$(D(e) \circ D(f))_{e-} = \max\{y; y \in D(e) \circ D(f) \text{ and } y \leqslant e\} = ef.$$

Hence, in both cases, $(e\varphi)(f\varphi) = (1, \{ef\}, D(ef)) = (ef)\varphi$. Finally, in the general case, we have $D(ef) \preccurlyeq D(e)$ and $D(ef) \preccurlyeq D(f)$. By the results proved above, $(e\varphi)((ef)\varphi) = (e(ef))\varphi = (ef)\varphi$ and $((ef)\varphi)(f\varphi) = ((ef)f)\varphi = (ef)\varphi$. Moreover, ef lies between e and f and so, since φ is an order isomorphism, $(ef)\varphi$ lies between $e\varphi$ and $f\varphi$. Therefore, by Lemma 1.2,

$$(e\varphi)(f\varphi) = (e\varphi)((ef)\varphi)(f\varphi) = ((e\varphi)((ef)\varphi))(((ef)\varphi)(f\varphi)) = (ef)\varphi.$$

(2) *Let $a = (\alpha, A_1, A_2) \in S'$. If $\Delta(D(A_1))$ is of L-type and $a = (\alpha, \{a_1\}, D(A_2))$, then we put $x = (\alpha^{-1}, \{a_2\}, D(a_1))$ with $a_2 \in A_2$. If*

$\Delta(D(A_1))$ is not of L-type and $a = (\alpha, D(A_1), \{a_2\})$, then we put $x = (\alpha^{-1}, D(a_2), \{a_1\})$ with $a_1 \in A_1$. Then, in both cases, we have $x \in S'$, $ax = a_1\varphi$, $xa = a_2\varphi$, and x is an inverse of a, where φ is the isomorphism of E' onto E defined in (1).

For the proof we consider only the case where $\Delta(D(A_1))$ is of L-type. In (9) of the proof of Theorem 5.1, it was shown that $x \in S'$, $ax = (1, \{a_1\}, D(a_1))$, and $axa = a$. Similarly we can show that $xa = (1, \{a_2\}, D(a_2))$ and $xax = x$. Hence, by (10) of the proof of Theorem 5.1, we have $ax, xa \in E$ and, by definition, $ax = a_1\varphi$ and $xa = a_2\varphi$. Moreover, since $axa = a$ and $xax = x$ the element x is an inverse of a.

(3) *Let $a = (\alpha, A_1, A_2)$ and $b = (\beta, B_1, B_2)$ be elements of S'. Then we have $a\sigma b$ if and only if $\alpha = \beta$.*

Let x be an element defined in (2). Then x is an inverse of a and the Γ-component of x is α^{-1}. Hence the Γ-component of xb is $\alpha^{-1}\beta$ and so, by (10) of the proof of Theorem 5.1, $xb \in E$ if and only if $\alpha = \beta$. Therefore, by Lemma 2.6 and the definition of σ, we have $a\sigma b$ if and only if $\alpha = \beta$.

(4) *Γ' is isomorphic as an ordered group with $\Gamma = S/\sigma$.*

Let ψ be the mapping of Γ into Γ' which maps $a\sigma^{\natural}$ to the Γ-component of a for each $a \in S'$. By (3), ψ is well-defined irrespective of the choice of $a \in S'$ and is one-to-one. Conversely, we take $\alpha \in \Gamma'$ arbitrarily. Then, by condition (ii) of Theorem 5.1, there exists a $\mathscr{D}_{E'}$-class F such that $\alpha \in \Gamma(F)$. If $\Delta(F)$ is of L-type, we put $a = (\alpha, \{f\}, F^{\alpha})$ with $f \in F$. If $\Delta(F)$ is not of L-type, we put $a = (\alpha, F, \{g\})$ with $g \in F^{\alpha}$. Then $a \in S'$ and $(a\sigma^{\natural})\psi = \alpha$. Hence ψ is a one-to-one mapping of Γ onto Γ'. Moreover, by definition, the Γ-component of the product of two elements in S' is equal to the product of Γ-components of these elements. Hence, for $a, b \in S'$, we have

$$((a\sigma^{\natural})\,\psi)\,((b\sigma^{\natural})\,\psi) = ((ab)\,\sigma^{\natural})\,\psi = ((a\sigma^{\natural})\,(b\sigma^{\natural}))\,\psi$$

and so ψ is a group isomorphism. Furthermore, if $a\sigma^{\natural} \leqslant b\sigma^{\natural}$, then for two σ-classes $a\sigma$ and $b\sigma$ there exist elements $a' \in a\sigma$ and $b' \in b\sigma$ such that $a' \leqslant b'$. But the Γ-components of a' and b' are $(a'\sigma^{\natural})\,\psi = (a\sigma^{\natural})\,\psi$ and $(b'\sigma^{\natural})\,\psi = (b\sigma^{\natural})\psi$, respectively. Hence, by definition, we have $(a\sigma^{\natural})\,\psi \leqslant (b\sigma^{\natural})\,\psi$. Since ψ is one-to-one $a\sigma^{\natural} < b\sigma^{\natural}$ implies $(a\sigma^{\natural})\,\psi < (b\sigma^{\natural})\,\psi$. Therefore conversely, if $(a\sigma^{\natural})\,\psi \leqslant (b\sigma^{\natural})\,\psi$, then $a\sigma^{\natural} \leqslant b\sigma^{\natural}$. Hence ψ is an ordered group isomorphism.

(5) *Let $a = (\alpha, A_1, A_2) \in S'$, $a_1 \in A_1$, and $a_2 \in A_2$. Then we have $a\mathscr{R}a_1\rho$ and $a\mathscr{L}a_2\varphi$.*

If $\Delta(D(A_1))$ is of L-type, we put $x = (\alpha^{-1}, \{a_2\}, D(a_1))$. If $\Delta(D(A_1))$ is not of L-type, we put $x = (\alpha^{-1}, D(a_2), \{a_1\})$. Then, by (2), we have $ax = a_1\varphi$, $xa = a_2\varphi$, $axa = a$, and $xax = x$. Hence $a\mathscr{R}ax = a_1\varphi$ and $a\mathscr{L}xa = a_2\varphi$.

(6) *Let* $a = (\alpha, A_1, A_2)$ *and* $b = (\beta, B_1, B_2)$ *be elements of* S'. *Then we have* $a\mathcal{R}b$ *if and only if* $A_1 = B_1$. *Also we have* $a\mathcal{L}b$ *if and only if* $A_2 = B_2$.

In this paper we prove only the first assertion. First we suppose that $a\mathcal{R}b$. We take $a_1 \in A_1$ and $b_1 \in B_1$ arbitrarily. Then, by (5), $a_1\varphi\mathcal{R}a\mathcal{R}b\mathcal{R}b_1\varphi$. Hence $(a_1\varphi)(b_1\varphi) = b_1\varphi$, $(b_1\varphi)(a_1\varphi) = a_1\varphi$ and so, since φ is an isomorphism, we have $a_1b_1 = b_1$ and $b_1a_1 = a_1$. Therefore $a_1\mathcal{R}b_1$ and, in particular, we have $D(A_1) = D(B_1)$. If $\Delta(D(A_1))$ is of L-type, then both A_1 and B_1 are one-element subsets and, since $a_1\mathcal{R}b_1$ we have $a_1 = b_1$. Hence $A_1 = \{a_1\} = \{b_1\} = B_1$. On the other hand, if $\Delta(D(A_1))$ is not of L-type, then $A_1 = D(A_1) = D(B_1) = B_1$.

(7) *For a* $\mathcal{D}_{E'}$*-class* F, $\Gamma(F)$ *coincides with that defined in Section 4.*

We fix $f \in F$ arbitrarily and suppose that $\alpha \in \Gamma(F)$. If $\Delta(F)$ is of L-type, we put $a = (\alpha, \{f\}, F^\alpha)$. If $\Delta(F)$ is not of L-type, we put $a = (\alpha, F, \{g\})$ with $g \in F^\alpha$. Then $a \in S'$ and, by (5), we have $a\mathcal{R}f\varphi$. On the other hand, we have $(a\sigma^\natural)\psi = \alpha$ and so $\alpha \in \{(a\sigma^\natural)\psi;\ a\mathcal{R}f\varphi\}$. Hence $\Gamma(F) \subseteq \{(a\sigma^\natural)\psi;\ a\mathcal{R}f\varphi\}$. Conversely, let $a\mathcal{R}f\varphi$ and let $a = (\alpha, A_1, A_2)$. We take $a_1 \in A_1$ arbitrarily. Then, by (5), we have $a_1\varphi\mathcal{R}a\mathcal{R}f\varphi$ and so $a_1\mathcal{R}f$. Hence $f \in D(A_1)$ and so $\alpha \in \Gamma(D(A_1)) = \Gamma(D(f)) = \Gamma(F)$. Hence $\Gamma(F) = \{(a\sigma^\natural)\psi;\ a\mathcal{R}f\varphi\}$. Identifying corresponding elements by φ and ψ, the set of the right-hand side is $\Gamma(F)$ defined in Section 4 and so the assertion has been obtained.

(8) *For a* $\mathcal{D}_{E'}$*-class* F *and* $\alpha \in \Gamma$ *such that* $\alpha \in \Gamma(F)$, F^α *coincides with that defined in Section 4.*

In fact, we choose $f \in F$ and $g \in F^\alpha$ arbitrarily. If $\Delta(F)$ is of L-type, we put $a = (\alpha, \{f\}, F^\alpha)$ and $x = (\alpha^{-1}, \{g\}, F)$. If $\Delta(F)$ is not of L-type, we put $a = (\alpha, F, \{g\})$ and $x = (\alpha^{-1}, F^\alpha, \{f\})$. Then a, $x \in S'$ and, by (2) and (5), we have in both cases $a\mathcal{R}f\varphi$, $(a\sigma^\natural)\psi = \alpha$, x is an inverse of a, and $xa = g\varphi$. Hence $D(xa) = D(g\varphi) = (D(g))\varphi = (F^\alpha)\varphi$. But $D(xa)$ is the F^α defined in Section 4. Thus the assertion has been obtained.

This completes the proof of Theorem 5.2.

References

1. Clifford, A. H. Totally ordered commutative semigroups. *Bull. Am. Math. Soc.* **64** (1958), 305-316.
2. Clifford, A. H. and Preston, G. B. "The Algebraic Theory of Semigroups, I." Am. Math. Soc., Math. Surveys, No. 7, 1961.
3. Dubreil, P. Contribution à la théorie des demi-groupes. *Mém. Acad. Sci. Inst. France* **63** (1941), 1-52.
4. Howie, J. M. and Lallement, G. Certain fundamental congruences on a regular semigroup. *Proc. Glasgow Math. Assoc.* **7** (1966), 145-159.
5. Munn, W. D. A class of irreducible matrix representation of an arbitrary inverse semigroup. *Proc. Glasgow Math. Assoc.* **5** (1961), 41-48.

6. SAITÔ, T. Ordered idempotent semigroups. *J. Math. Soc. Japan* **14** (1962), 150-169.
7. SAITÔ, T. Regular elements in an ordered semigroup. *Pacific J. Math.* **13** (1963), 263-295.
8. SAITÔ, T. Ordered completely regular semigroups. *Pacific J. Math.* **14** (1964), 295-308.
9. SAITÔ, T. Proper ordered inverse semigroups. *Pacific J. Math.* **15** (1965), 649-666.
10. TEISSIER, M. Sur les équivalences régulières dans les demi-groupes. *C.R. Acad. Sci. (Paris)* **232** (1951), 1987-1989.